13 Optimal Power Flow

13.1 INTRODUCTION

The optimal power flow of OPF has had a long history in its development. It was first discussed by Carpentier in 1962 (reference 1) and took a long time to become a successful algorithm that could be applied in everyday use. Current interest in the OPF centers around its ability to solve for the optimal solution that takes account of the security of the system.

In Chapter 3, we introduced the concept of economic dispatch. In the economic dispatch we had a single constraint which held the total generation to equal the total load plus losses. Thus, the statement of the economic dispatch problem results in a Lagrangian with just one constraint:

$$L = \sum F(P_i) + \lambda(P_{\text{load}} + P_{\text{losses}} - \sum P_i) \tag{13.1}$$

If we think about the single "generation equals load plus losses" constraint:

$$P_{\text{load}} + P_{\text{losses}} - \sum P_i = 0 \tag{13.2}$$

we realize that what it is actually saying is that the generation must obey the same conditions as expressed in a power flow—with the condition that the entire power flow is reduced to one simple equality constraint. There is good reason, as we shall see shortly, to state the economic dispatch calculation in terms of the generation costs, and the entire set of equations needed for the power flow itself as constraints. The power flow equations were introduced in Chapter 4. This formulation is called an optimal power flow.

We can solve the OPF for the minimum generation cost (as in Chapter 3) and require that the optimization calculation also balance the entire power flow—at the same time. Note also that the objective function can take different forms other than minimizing the generation cost. It is common to express the OPF as a minimization of the electrical losses in the transmission system, or to express it as the minimum shift of generation and other controls from an optimum operating point. We could even allow the adjustment of loads in order to determine the minimum load shedding schedule under emergency conditions. Regardless of the objective function, however, an OPF must solve so that the entire set of power constraints are present and satisfied at the solution.

Why set up the generation dispatch calculation as an OPF?.

1. If the entire set of power flow equations are solved simultaneously with the generation cost minimization, the representation of incremental losses is exact. Further, with an objective function that minimizes the losses themselves, the power flow equations are quite necessary.

2. The economic dispatch solutions in Chapter 3 only observed the generation limits $P_i^- \leq P_i \leq P_i^+$. With all of the power flow constraints included in the formulation, many more of the power system limits can be included. These include limits on the generator reactive power, $Q_i^- \leq Q_i \leq Q_i^+$, limits on the voltage magnitude at generation and load buses, $|E_i|^- \leq |E_i| \leq |E_i|^+$, and flows on transmission lines or transformers expressed in either MW, amperes or MVA (e.g. $\text{MVA}_{ij}^- \leq \text{MVA}_{ij} \leq \text{MVA}_{ij}^+$). This set of operating constraints now allows the user to guarantee that the dispatch of generation does not, in fact, force the transmission system into violating a limit, which might put it in danger of being damaged.

3. The OPF can also include constraints that represent operation of the system after contingency outages. These "security constraints" allow the OPF to dispatch the system in a defensive manner. That is, the OPF now forces the system to be operated so that if a contingency happened, the resulting voltages and flows would still be within limit. Thus, constraints such as the following might be incorporated:

$$|E_k|^- \leq |E_k| \text{ (with line } nm \text{ out)} \leq |E_k|^+ \tag{13.3}$$

$$\text{MVA}_{ij}^- \leq \text{MVA}_{ij} \text{ (with line } nm \text{ out)} \leq \text{MVA}_{ij}^+ \tag{13.4}$$

which implies that the OPF would prevent the post-contingency voltage on bus k or the post-contingency flow on line ij from exceeding their limits for an outage of line nm. This special type of OPF is called a "security-constrained OPF," or SCOPF.

4. In the dispatch calculation developed in Chapter 3, the only adjustable variables were the generator MW outputs themselves. In the OPF, there are many more adjustable or "control" variables that be be specified. A partial list of such variables would include:

- Generator voltage.
- LTC transformer tap position.
- Phase shift transformer tap position.
- Switched capacitor settings.
- Reactive injection for a static VAR compensator.
- Load shedding.
- DC line flow.

Thus, the OPF gives us a framework to have many control variables adjusted in the effort to optimize the operation of the transmission system.

5. The ability to use different objective functions provides a very flexible analytical tool.

Given this flexibility, the OPF has many applications including:

1. The calculation of the optimum generation pattern, as well as all control variables, to achieve the minimum cost of generation together with meeting the transmission system limitations.
2. Using either the current state of the power system or a short-term load forecast, the OPF can be set up to provide a "*preventative dispatch*" if security constraints are incorporated.
3. In an emergency, that is when some component of the system is overloaded or a bus is experiencing a voltage violation, the OPF can provide a "corrective dispatch" which tells the operators of the system what adjustments to make to relieve the overload or voltage violation.
4. The OPF can be used periodically to find the optimum setting for generation voltages, transformer taps and switched capacitors or static VAR compensators (sometimes called "voltage–VAR" optimization).
5. The OPF is routinely used in planning studies to determine the maximum stress that a planned transmission system can withstand. For example, the OPF can calculate the maximum power that can safely be transferred from one area of the network to another.
6. The OPF can be used in economic analyses of the power system by providing "*bus incremental costs*" (BICs). The BICs are useful to determine the marginal cost of power at any bus in the system. Similarly, the OPF can be used to calculate the incremental or marginal cost of transmitting power from one outside company—through its system—to another outside company.

13.2 SOLUTION OF THE OPTIMAL POWER FLOW

The optimal power flow is a very large and very difficult mathematical programming problem. Almost every mathematical programming approach that can be applied to this problem has been attempted and it has taken developers many decades to develop computer codes that will solve the OPF problem reliably.

Chapter 3 introduced the concept of the lambda-iteration methods, the gradient method and Newton's method. We shall review all of these here and introduce two new techniques, the linear programming (LP) method and the interior point method. The attributes of these methods are summarized next.

- **Lambda iteration method:** Losses may be represented by a [B] matrix, or the penalty factors may be calculated outside by a power flow. This forms the basis of many standard on-line economic dispatch programs.
- **Gradient methods:** Gradient methods are slow in convergence and are difficult to solve in the presence of inequality constraints.
- **Newton's method:** Very fast convergence, but may give problems with inequality constraints.
- **Linear programming method (LPOPF):** One of the fully developed methods now in common use. Easily handles inequality constraints. Nonlinear objective functions and constraints handled by linearization.
- **Interior point method:** Another of the fully developed and widely used methods for OPF. Easily handles inequality constraints.

We introduced and analyzed the lambda-iteration method in Chapter 3. This method forms the basis of standard on-line economic dispatch codes. The technique works well and can be made to run very fast. It overlooks any constraints on the transmission system and does not produce a dispatch of the generation that will avoid overloads, voltage limit violations, or security constraint violations.

We shall derive the gradient method using the same mathematics used in Chapter 3, only with various advanced models of the transmission system instead of the "load plus losses equals generation" constraint used in Chapter 3. It is then a simple step to go on to develop the Newton's method applied with these same constraints. Finally, the LPOPF and interior point methods are presented.

The objective function in the OPF is usually minimized. In some cases, such as power transfers, it may be maximized. We shall designate the objective function as f. The equations that guarantee that the power flow constraints are met will be designated as

$$\mathbf{g}(\mathbf{z}) = 0 \tag{13.5}$$

Note that here we shall only be concerned with a variable vector $\mathbf{z}$. This vector contains the adjustable controls, the bus voltage magnitudes, and phase angles, as well as the fixed parameters of the system, Later, we shall break the variables up into sets of state variables, control variables, and fixed parameters.

The OPF can also solve for an optimal solution with inequality constraints on dependent variables, such as line MVA flows. These will be designated

$$\mathbf{h}^- \le \mathbf{h}(\mathbf{z}) \le \mathbf{h}^+ \tag{13.6}$$

In addition, limits may be placed directly on state variables or control variables:

$$\mathbf{z}^- \le \mathbf{z} \le \mathbf{z}^+ \tag{13.7}$$

The OPF problem then consists of minimizing (or maximizing) the objective function, subject to the equality constraints, the inequality constraints, and the state and control variable limits.

The developments and illustrative examples in this chapter concentrate (but not exclusively) on the LPOPF technique. The method is widely used and only requires an AC or DC power flow program, plus a suitable LP package for solving illustrative examples and (homework) problems.

13.2.1 The Gradient Method

In this section, we shall consider the objective function to be total cost of generation (later examples will demonstrate how other objectives can be used). The objective function to be minimized is:

$$\sum_{\text{all gen.}} F_i(P_i)$$

where the sum extends to all generators on the power system, including the generator at the reference bus.

We shall start out defining the unknown or state vector $\mathbf{x}$ as:

$$\mathbf{x} = \begin{bmatrix} \left.\begin{matrix} \theta_i \\ |E_i| \end{matrix}\right\} \text{on each } PQ \text{ bus} \\ \theta_i \quad \text{on each } PV \text{ bus} \end{bmatrix} \tag{13.8}$$

another vector, $\mathbf{y}$, is defined as:

$$\mathbf{y} = \begin{bmatrix} \left.\begin{matrix} \theta_k \\ |E_k| \end{matrix}\right\} \text{on the reference bus} \\ \left.\begin{matrix} P_k^{\text{net}} \\ Q_k^{\text{net}} \end{matrix}\right\} \text{on each } PQ \text{ bus} \\ \left.\begin{matrix} P_k^{\text{net}} \\ |E_k|^{\text{sch}} \end{matrix}\right\} \text{on each } QV \text{ bus} \end{bmatrix} \tag{13.9}$$

Note that the vector $\mathbf{y}$ is made up of all of the parameters that must be specified. Some of these parameters are adjustable (for example, the generator output, P_k^{net}, and the generator bus voltage). Some of the parameters are fixed, as far as the OPF calculation is concerned, such as the P and Q at each load bus. To make this distinction, we shall divide the $\mathbf{y}$ vector up into two parts, $\mathbf{u}$ and $\mathbf{p}$:

$$\mathbf{y} = \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} \tag{13.10}$$

where **u** represents the vector of control or adjustable variables, and **p** represents the fixed or constant variables. Note also that we are only representing equality constraints at this point.

Finally, we shall define a set of m equations that govern the power flow:

$$\mathbf{g}(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} P_i(|\mathbf{E}|, \theta) - P_i^{\text{net}} \\ Q_i(|\mathbf{E}|, \theta) - Q_i^{\text{net}} \\ P_k(|\mathbf{E}|, \theta) - P_k^{\text{net}} \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{for each } PQ \text{ (load) bus } i \\ \text{for each } PV \text{ (gen.) bus } k, \text{ not including the reference bus} \end{matrix} \tag{13.11}$$

Note that these equations are the same bus equations as shown in Chapter 4 for the Newton power flow (Eq. 4.18).

We must recognize that the reference-bus power generation is not an independent variable. That is, the reference-bus generation always changes to balance the power flow; we cannot specify it at the beginning of the calculation. We wish to express the cost or objective function as a function of the control variables and of the state variables. We do this by dividing the cost function as follows:

$$\text{cost} = \sum_{\text{gen}} F_i(P_i) + F_{\text{ref}}(P_{\text{ref}}) \tag{13.12}$$

where the first summation does not include the reference bus, The P_i are all independent, controlled variables whereas P_{ref} is a dependent variable. We say that the P_i are in the vector **u** and the P_{ref} is a function of the network voltages and angles:

$$P_{\text{ref}} = P_{\text{ref}}(|\mathbf{E}|, \theta) \tag{13.13}$$

then the cost function becomes:

$$\sum_{\text{gen}} F_i(P_i) + F_{\text{ref}}[P_{\text{ref}}(|\mathbf{E}|, \theta)] = \mathrm{f}(\mathbf{x}, \mathbf{u}) \tag{13.14}$$

We can now set up the Lagrange equation for the OPF as follows:

$$\mathscr{L}(\mathbf{x}, \mathbf{u}, \mathbf{p}) = \mathrm{f}(\mathbf{x}, \mathbf{u}) + \lambda^t \mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{p}) \tag{13.15}$$

where

x = vector of state variables

u = vector of control variables

p = vector of fixed parameters

λ = vector of Lagrange multipliers

g = set of equality constraints representing the power flow equations

f = the objective function

This Lagrange equation is perhaps better seen when written as:

$$\mathscr{L}(\mathbf{x}, \mathbf{u}, \mathbf{p}) = \sum_{\text{gen}} F_i(P_i) + F_{\text{ref}}[P_{\text{ref}}(|\mathbf{E}|, \boldsymbol{\theta})] + [\lambda_1 \lambda_2, \ldots, \lambda_m] \begin{bmatrix} P_i(|\mathbf{E}|, \boldsymbol{\theta}) - P_i^{\text{net}} \\ Q_i(|\mathbf{E}|, \boldsymbol{\theta}) - Q_i^{\text{net}} \\ P_k(|\mathbf{E}|, \boldsymbol{\theta}) - P_k^{\text{net}} \\ \vdots \end{bmatrix} \tag{13.16}$$

We now have a Lagrange function that has a single objective function and m Lagrange multipliers, one for each of the m power flow equations.

To minimize the cost function, subject to the constraints, we set the gradient of the Lagrange function to zero:

$$\nabla \mathscr{L} = 0 \tag{13.17}$$

To do this, we break up the gradient vector into three parts corresponding to the variables $\mathbf{x}$, $\mathbf{u}$, and $\boldsymbol{\lambda}$:

$$\nabla \mathscr{L}_x = \frac{\partial \mathscr{L}}{\partial \mathbf{x}} = \frac{\partial \mathrm{f}}{\partial \mathbf{x}} + \left[\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right]^T \boldsymbol{\lambda} \tag{13.18}$$

$$\nabla \mathscr{L}_u = \frac{\partial \mathscr{L}}{\partial \mathbf{u}} = \frac{\partial \mathrm{f}}{\partial \mathbf{u}} + \left[\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\right]^T \boldsymbol{\lambda} \tag{13.19}$$

$$\nabla \mathscr{L}_\lambda = \frac{\partial \mathscr{L}}{\partial \boldsymbol{\lambda}} = \mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{p}) \tag{13.20}$$

Some discussion of the three gradient equations above is in order. First, Eq. 13.18 consists of a vector of derivatives of the objective function with respect to the state variables, x. Since the objective function itself is not a function of the state variable *except for the reference bus*, this becomes:

$$\frac{\partial \mathrm{f}}{\partial \mathbf{x}} = \begin{bmatrix} \dfrac{\partial}{\partial P_{\text{ref}}} F_{\text{ref}}(P_{\text{ref}}) \dfrac{\partial P_{\text{ref}}}{\partial \theta_1} \\ \dfrac{\partial}{\partial P_{\text{ref}}} F_{\text{ref}}(P_{\text{ref}}) \dfrac{\partial P_{\text{ref}}}{\partial |E_1|} \\ \vdots \end{bmatrix} \tag{13.21}$$

The $[\partial \mathbf{g}/\partial \mathbf{x}]$ term in Eq. 13.18 actually is the Jacobian matrix for the Newton power flow, which was developed in Chapter 4. That is:

$$\left[\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right] = \begin{bmatrix} \frac{\partial P_1}{\partial \theta_1} & \frac{\partial P_1}{\partial |E_1|} & \frac{\partial P_1}{\partial \theta_2} & \frac{\partial P_1}{\partial |E_2|} & \cdots \\ \frac{\partial Q_1}{\partial \theta_1} & \frac{\partial Q_1}{\partial |E_1|} & \frac{\partial Q_1}{\partial \theta_2} & \frac{\partial Q_1}{\partial |E_2|} & \cdots \\ \frac{\partial P_2}{\partial \theta_1} & \frac{\partial P_2}{\partial |E_1|} & & & \cdots \\ \frac{\partial Q_2}{\partial \theta_1} & \frac{\partial Q_2}{\partial |E_1|} & & & \cdots \\ \vdots & & & & \vdots \end{bmatrix} \tag{13.22}$$

Note that this matrix must be transposed for use in Eq. 13.18.

Equation 13.19 is the gradient of the Lagrange function with respect to the control variables. Here the vector $\partial \mathbf{f}/\partial \mathbf{u}$ is a vector of derivatives of the objective function with respect to the control variables:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \begin{bmatrix} \frac{\partial}{\partial P_1} F_1(P_1) \\ \frac{\partial}{\partial P_2} F_2(P_2) \\ \vdots \end{bmatrix} \tag{13.23}$$

The other term in Eq. 13. 19, $[\partial \mathbf{g}/\partial \mathbf{u}]$, actually consists of a matrix of all zeros with some -1 terms on the diagonals, which correspond to equations in $\mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{p})$ where a control variable is present. Finally, Eq. 13.20 consists simply of the power flow equations themselves.

The solution of the gradient method of OPF is as follows:

1. Given a set of fixed parameters $\mathbf{p}$, assume a starting set of control variables $\mathbf{u}$.
2. Solve a power flow. This guarantees that Eq. 13.20 is satisfied.
3. Solve Eq. 13.19 for lambda;

$$\lambda = -\left[\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right]^{T^{-1}} \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \tag{13.24}$$

4. Substitute λ into Eq. 13.18 to get the gradient of $\mathscr{L}$ with respect to the control variables.

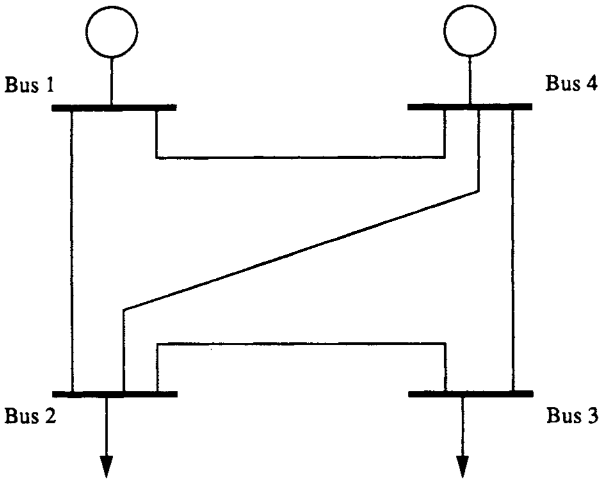

FIG. 13.1 Four-bus system for Example 13A.

The gradient will give the direction of maximum increase in the cost function as a function of the adjustments in each of the **u** variables. Since we wish to decrease the objective function, we shall move in the direction of the negative of the gradient. The gradient method gives no indication how far along the negative gradient direction we should move. Assuming that a distance is picked that reduces the objective, one must start at step 2 above, and repeat steps 2, 3, and 4 over and over until the gradient itself becomes sufficiently close to the zero vector, indicating that all conditions for the optimum have been reached.

EXAMPLE 13A

The following is a very simple example presented to show the meaning of each of the elements in the gradient equations. Example 13B will be a more practical example of the gradient method.

The four-bus system in Figure 13.1 will be modeled with a DC power flow. The following are known:

P_2, P_3, and $\theta_4 = 0$

Line reactances: x_{12}, x_{14}, x_{24}, x_{23}, and x_{34}

Cost functions: $F_1(P_1)$ and $F_4(P_4)$

All $|E|$ values are fixed at 1.0 per unit volts

The only independent control variable in this problem is the generator output P_1, or:

$$u = P_1 \qquad (13.25)$$

The state variables are θ_1, θ_2, and θ_3, or:

$$\mathbf{x} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \tag{13.26}$$

We wish to minimize the total generation cost while maintaining a solved DC power flow for the network. To do this with the gradient method we form the Lagrangian:

$$\mathscr{L} = F_1(P_1) + F_4[P_4(\theta_1 \ldots \theta_4)] + [\lambda_1 \quad \lambda_2 \quad \lambda_3] \begin{bmatrix} P_1(\theta_1 \ldots \theta_4) - P_1 \\ P_2(\theta_1 \ldots \theta_4) - P_2 \\ P_3(\theta_1 \ldots \theta_4) - P_3 \end{bmatrix} \tag{13.27}$$

In terms of the equations presented earlier:

$$\mathrm{f}(\mathbf{x}, \mathbf{u}) = F_1(P_1) + F_4[P_4(\theta_1 \ldots \theta_4)] \tag{13.28}$$

$$g(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} P_1(\theta_1 \ldots \theta_4) - P_1 \\ P_2(\theta_1 \ldots \theta_4) - P_2 \\ P_3(\theta_1 \ldots \theta_4) - P_3 \end{bmatrix} \tag{13.29}$$

Note that in $\mathbf{g}(\mathbf{x}, \mathbf{u})$, the P_1 is the control variable and P_2 and P_3 are fixed.

We shall now expand $\mathbf{g}(\mathbf{x}, \mathbf{u})$ as follows:

$$\mathbf{g}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} P_1(\theta_1 \ldots \theta_4) - P_1 \\ P_2(\theta_1 \ldots \theta_4) - P_2 \\ P_3(\theta_1 \ldots \theta_4) - P_3 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{x_{12}}(\theta_1 - \theta_2) + \dfrac{1}{x_{14}}(\theta_1 - \theta_{24}) - P_1 \\ \vdots \\ \; \end{bmatrix} \tag{13.30}$$

The result is:

$$\mathbf{g}(\mathbf{x}, \mathbf{u}) = [B'] \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} - \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{13.31}$$

and the Lagrange function becomes:

$$F_1(P_1) + F_4[P_4(\theta_1 \ldots \theta_4)] + [\lambda_1 \quad \lambda_2 \quad \lambda_3] \left([B'] \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} - \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \right) \tag{13.32}$$

We now proceed to develop the three gradient components:

$$\nabla \mathscr{L}_{\lambda} = \mathbf{g}(\mathbf{x}, \mathbf{u}) = 0 \tag{13.33}$$

which simply says that we need to start by always maintaining the DC power flow:

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = [B']^{-1} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{13.34}$$

The next component:

$$\Delta \mathscr{L}_x = \begin{bmatrix} \dfrac{\partial \mathscr{L}}{\partial \theta_1} \\ \dfrac{\partial \mathscr{L}}{\partial \theta_2} \\ \dfrac{\partial \mathscr{L}}{\partial \theta_3} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial F_4}{\partial P_4} \dfrac{\partial P_4}{\partial \theta_1} \\ \dfrac{\partial F_4}{\partial P_4} \dfrac{\partial P_4}{\partial \theta_2} \\ \dfrac{\partial F_4}{\partial P_4} \dfrac{\partial P_4}{\partial \theta_3} \end{bmatrix} + [B']^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = 0 \tag{13.35}$$

This can be used to solve the vector of Lagrange multipliers:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = (-1)[B']^{T^{-1}} \begin{bmatrix} \dfrac{\partial P_4}{\partial \theta_1} \\ \dfrac{\partial P_4}{\partial \theta_2} \\ \dfrac{\partial P_4}{\partial \theta_3} \end{bmatrix} \dfrac{\partial F_4}{\partial P_4} \tag{13.36}$$

where

$$\begin{bmatrix} \dfrac{\partial P_4}{\partial \theta_1} \\ \dfrac{\partial P_4}{\partial \theta_2} \\ \dfrac{\partial P_4}{\partial \theta_3} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{x_{14}} \\ -\dfrac{1}{x_{24}} \\ -\dfrac{1}{x_{34}} \end{bmatrix} \tag{13.37}$$

It can be easily demonstrated that:

$$[B']^{T^{-1}}\begin{bmatrix}\dfrac{\partial P_4}{\partial \theta_1}\\ \dfrac{\partial P_4}{\partial \theta_2}\\ \dfrac{\partial P_4}{\partial \theta_3}\end{bmatrix} = \begin{bmatrix}-1\\-1\\-1\end{bmatrix} \tag{13.38}$$

so that

$$\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}\frac{\partial F_4}{\partial P_4} \tag{13.39}$$

Finally,

$$\frac{\partial \mathbf{g}}{\partial u} = \begin{bmatrix}-1\\0\\0\end{bmatrix} \tag{13.40}$$

and

$$\nabla \mathscr{L}_u = \frac{\partial F_1}{\partial P_1} + \frac{\partial \mathbf{g}^T}{\partial u}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix} = \frac{\partial F_1}{\partial P_1} + [-1 \quad 0 \quad 0]\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\frac{\partial F_4}{\partial P_4}\right) = \frac{\partial F_1}{\partial P_1} - \frac{\partial F_4}{\partial P_4} \tag{13.41}$$

the gradient with respect to the control variable is zero when the two incremental costs are equal, which is the common economic dispatch criterion (assuming neither generator is at a limit). Since the DC power flow represents a linear lossless system, the result simply confirms that the gradient method will produce a result that is the same as economic dispatch.

EXAMPLE 13B

In this example, we shall minimize the real power losses (MW losses) on the three-bus AC system in Figure 13.2. To work this example, the student must be able to run an AC power flow on the three-bus system. (This example is taken from reference 4.)

Given the three-bus network shown in Figure 13.2, where

$$P_3 + jQ_3 = 2.0 + j1.0 \text{ per unit}$$

and

$$P_2 = 1.7 \text{ per unit}$$

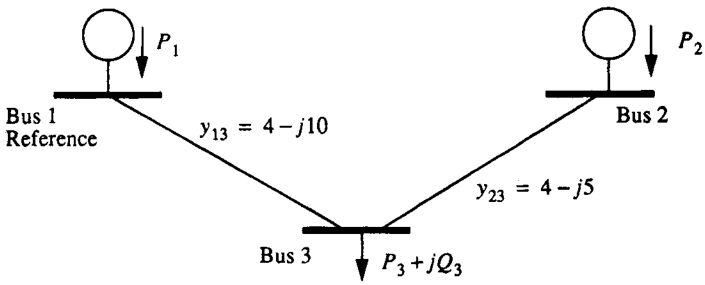

FIG. 13.2 Three-bus example for Example 13B.

In this problem, the generation at bus 2 will be fixed, *the only control variables will be the voltage magnitude at buses 1 and 2.* That is

$$\mathbf{u} = \begin{bmatrix} |E_1| \\ |E_2| \end{bmatrix} \tag{13.42}$$

The state variables will be the phase angles at buses 2 and 3 and the voltage at bus 3:

$$\mathbf{x} = \begin{bmatrix} \theta_2 \\ \theta_3 \\ |E_3| \end{bmatrix} \tag{13.43}$$

The fixed parameters are

$$\mathbf{p} = \begin{bmatrix} \theta_1 \\ P_2 \\ P_3 \\ Q_3 \end{bmatrix} \tag{13.44}$$

We shall solve for the minimum losses using the gradient method. This requires that we solve, repeatedly, the following:

$$\nabla \mathscr{L}_u = \begin{bmatrix} \dfrac{\partial P_{\text{losses}}}{\partial |E_1|} \\ \dfrac{\partial P_{\text{losses}}}{\partial |E_2|} \end{bmatrix} \tag{13.45}$$

Starting at an initial set of voltages:

$$\begin{bmatrix} |E_1| \\ |E_2| \end{bmatrix}^0 = \begin{bmatrix} 1.1 \\ 0.9 \end{bmatrix} \tag{13.46}$$

we proceed using

$$\begin{bmatrix} |E_1| \\ |E_2| \end{bmatrix}^1 = \begin{bmatrix} |E_1| \\ |E_2| \end{bmatrix}^0 + (-1)(\nabla \mathscr{L}_u)\alpha \tag{13.47}$$

where α was set to 0.03 after several trials.

As previously:

$$\mathbf{g}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} P_2(\mathbf{E}|, \boldsymbol{\theta}) - P_2 \\ P_3(|\mathbf{E}|, \boldsymbol{\theta}) - P_3 \\ Q_3(|\mathbf{E}|, \boldsymbol{\theta}) - Q_3 \end{bmatrix} \tag{13.48}$$

where the above represents the AC power flow equations as shown in Chapter 4. When we take the derivative,

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial P_2}{\partial \theta_2} & \frac{\partial P_2}{\partial \theta_3} & \frac{\partial P_2}{\partial |E_3|} \\ \frac{\partial P_3}{\partial \theta_2} & \frac{\partial P_3}{\partial \theta_3} & \frac{\partial P_3}{\partial |E_3|} \\ \frac{\partial Q_3}{\partial \theta_2} & \frac{\partial Q_3}{\partial \theta_3} & \frac{\partial Q_3}{\partial |E_3|} \end{bmatrix} \tag{13.49}$$

these derivatives are calculated as shown in Chapter 4, Eq. 4.22 and the above represents the Jacobian matrix that would be used in the Newton power flow solution to this network. Similarly:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{u}} = \begin{bmatrix} \frac{\partial P_2}{\partial |E_1|} & \frac{\partial P_2}{\partial |E_2|} \\ \frac{\partial P_3}{\partial |E_1|} & \frac{\partial P_3}{\partial |E_2|} \\ \frac{\partial Q_3}{\partial |E_1|} & \frac{\partial Q_3}{\partial |E_2|} \end{bmatrix} \tag{13.50}$$

One special note, the objective function, P_{losses} can be expressed in two different ways. The first is simply to write out the losses as:

$$P_{\text{losses}} = \text{Re}\left(\sum_{\textit{both}\ \text{lines}} I^2 R \right) \tag{13.51}$$

or one can use the simple observation that since P_2 and P_3 are fixed, any change in the losses due to adjustments of V_1 and V_2 will be directly reflected in changes in P_1. That is, $\Delta P_{\text{losses}} = \Delta P_1$ and

$$\frac{\partial P_{\text{losses}}}{\partial \mathbf{x}} = \frac{\partial P_1}{\partial \mathbf{x}} \tag{13.52}$$

We shall use the second form of the objective so that

$$\text{f} = P_1 \tag{13.53}$$

and then:

$$\frac{\partial \text{f}}{\partial \mathbf{x}} = \begin{bmatrix} \dfrac{\partial P_1}{\partial \theta_2} \\ \dfrac{\partial P_1}{\partial \theta_3} \\ \dfrac{\partial P_1}{\partial |E_3|} \end{bmatrix} \tag{13.54}$$

The solution to the first AC power flow, with

$$\begin{bmatrix} |E_1| \\ |E_2| \end{bmatrix}^0 = \begin{bmatrix} 1.1 \\ 0.9 \end{bmatrix} \tag{13.55}$$

gives per unit losses of 0.3906 (39.06 MW losses on 100-MVA base). The reference-bus power, P_1, is 0.6906 per unit MW. Taking this solved power flow as the starting point, we have:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}} = \begin{bmatrix} 8.14 & 8.14 & 1.54 \\ 6.96 & 12.0 & 3.85 \\ -4.5 & -7.85 & 10.0 \end{bmatrix} \tag{13.56}$$

$$\frac{\partial \text{f}}{\partial \mathbf{x}} = \begin{bmatrix} 0 \\ 4.36 \\ 4.14 \end{bmatrix} \tag{13.57}$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = (-1)\begin{bmatrix} \dfrac{\partial \mathbf{g}}{\partial \mathbf{x}} \end{bmatrix}^{T^{-1}} \frac{\partial \text{f}}{\partial \mathbf{x}} = \begin{bmatrix} 0.743 \\ -0.98 \\ -0.154 \end{bmatrix} \tag{13.58}$$

$$\frac{\partial f}{\partial \mathbf{u}} = \begin{bmatrix} \frac{\partial P_1}{\partial V_1} \\ \frac{\partial P_1}{\partial V_2} \end{bmatrix} = \begin{bmatrix} 5.533 \\ 0 \end{bmatrix} \tag{13.59}$$

$$\left[\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\right]^T = \begin{bmatrix} 0 & 3.354 & 5.0 \\ 4.94 & 4.5 & 6.96 \end{bmatrix} \tag{13.60}$$

Then,

$$\Delta \mathscr{L}_u = \frac{\partial f}{\partial \mathbf{u}} + \left[\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\right]^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 2.25 \\ -1.78 \end{bmatrix} \tag{13.61}$$

and, with $\alpha = 0.03$, we obtain a new set of voltages:

$$\begin{bmatrix} |E_1| \\ |E_2| \end{bmatrix}^1 = \begin{bmatrix} 1.1 \\ 0.9 \end{bmatrix} - \begin{bmatrix} 2.25 \\ -1.787 \end{bmatrix} 0.03 = \begin{bmatrix} 0.95 \\ 1.03 \end{bmatrix} \tag{13.62}$$

This represents the new control variable settings that must be fed back to the AC power flow.

The new AC power flow, with the above new voltages, results in $P_{\text{losses}} = 0.2380$ per unit and the generation at the reference bus of $P_1 = 0.5380$. Another iteration of the gradient calculation yields $P_{\text{losses}} = 0.2680$ per unit for a controls setting of:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0.86 \\ 0.86 \end{bmatrix} \tag{13.63}$$

Note that for this simple problem, the gradient is able to find a reduction in losses after the first iteration, but the next iteration caused the losses to increase. Eventually, it will need tuning, in the form of additional adjustments to the value of α, so that it will not simply oscillate around a minimum. Further, we never specified any voltage limits for V_1 and V_2. As we reduce losses, we may very well run into voltage limits on buses 1 or 2, or both. Here, the gradient method loses whatever simplicity it has and tends to become unmanageable. This would further be the case if we were to place a limit on V_3, which would be a functional inequality and would be very difficult to express in the gradient formulation we have used.

13.2.2 Newton's Method

The problems with the gradient method lie mainly in the fact that the direction of the gradient must be changed quite often and this leads to a very slow convergence. To speed up this convergence, we can use Newton's method, where we take the derivative of the gradient with respect to **x**, **u**, and λ. Then, the optimal solution becomes:

$$\begin{bmatrix} \Delta \mathbf{x} \\ \Delta \mathbf{u} \\ \Delta \lambda \end{bmatrix} = -\begin{bmatrix} \frac{\partial}{\partial \mathbf{x}} \nabla \mathscr{L}_x & \frac{\partial}{\partial \mathbf{u}} \nabla \mathscr{L}_x & \frac{\partial}{\partial \lambda} \nabla \mathscr{L}_x \\ \frac{\partial}{\partial \mathbf{x}} \nabla \mathscr{L}_u & \frac{\partial}{\partial \mathbf{u}} \nabla \mathscr{L}_u & \frac{\partial}{\partial \lambda} \nabla \mathscr{L}_u \\ \frac{\partial}{\partial \mathbf{x}} \nabla \mathscr{L}_\lambda & \frac{\partial}{\partial \mathbf{u}} \nabla \mathscr{L}_\lambda & \frac{\partial}{\partial \lambda} \nabla \mathscr{L}_\lambda \end{bmatrix}^{-1} \begin{bmatrix} \nabla \mathscr{L}_x \\ \nabla \mathscr{L}_u \\ \nabla \mathscr{L}_\lambda \end{bmatrix} \tag{13.64}$$

The form of Eq. 13.22 is essentially the same as that derived in Section 3.5 on Newton's method. This matrix equation is a very formidable undertaking to compute and manipulate. It is extremely sparse and requires special sparsity logic.

Handling inequality constraints is very difficult in either gradient or Newton approaches. The usual method is to form a constraint "penalty" function as follows. Suppose the voltage at a bus must meet limits:

$$|E_i|^{\min} \leq |E_i| \leq |E_i|^{\max} \tag{13.65}$$

It is possible to enforce this constraint by inventing the following exterior penalty functions:

$$h(|E_i|) = \begin{bmatrix} K(|E_i| - |E_i|^{\min})^2 \\ 0 \\ K(|E_i|^{\max} - |E_i|)^2 \end{bmatrix} \begin{matrix} \text{for } |E_i| < |E_i|^{\min} \\ \text{for } E \text{ within limits} \\ \text{for } |E_i| > |E_i|^{\max} \end{matrix} \tag{13.66}$$

This penalty function is shown in Figure 13.3.

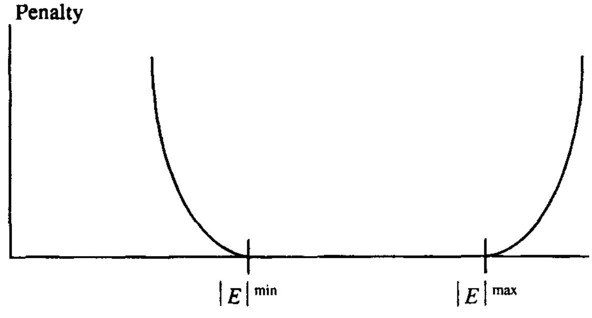

FIG. 13.3 Exterior penalty functions for voltage magnitude violations.

To solve the OPF with the voltage inequality constraint, we add the penalty function to the objective function, f. The resulting function will be large if the voltage is outside its limit, and thus the OPF will try to force it within its limits as it minimizes the objective.

Since Newton's method has the second derivative information built into it, it does not have great difficulty in converging and it can handle the inequality constraints as well. The difficulty with Newton's method arises in the fact that near the limit the penalty is small, so that the optimal solution will tend to allow the variable, a voltage in the example above, to float over its limit. The seemingly simple tuning procedure of raising the value of K may eventually cause the matrices to become ill-conditioned and the method fails. When there are few limits to be concerned with and the objective function is "shallow," that is, the variability of f with adjustments in the control variables is very low, Newton's method is the best method to use.

References 5–7 give examples of the development of Newton's method to solve the full AC OPF.

13.3 LINEAR SENSITIVITY ANALYSIS

Before continuing with the discussion of the linear programming and interior OPF methods, we shall develop the concept of linear sensitivity analysis. Linear sensitivity coefficients give an indication of the change in one system quantity (e.g., MW flow, MVA flow, bus voltage, etc.) as another quantity is varied (e.g., generator MW output, transformer tap position, etc.) These linear relationships are essential for the application of linear programming. Note that as the adjustable variable is changed, we assume that the power system reacts so as to keep all of the power flow equations solved. As such, linear sensitivity coefficients can be expressed as partial derivatives for example:

$$\frac{\partial \text{MVA flow}_{ij}}{\partial \text{MW gen}_k}$$

shows the sensitivity of the flow (MVA) on line (i to j) with respect to the power generated at bus k.

Some sensitivity coefficients may change rapidly as the adjustment is made and the power flow conditions are updated. This is because some system quantities vary in a nonlinear relationship with the adjustment and resolution of the power flow equations. This is especially true for quantities that have to do with voltage and MVAR flows. Sensitivities such as the variation of MW flow with respect to a change in generator MW output are rather linear across a wide range of adjustments and lead to the usefulness of the DC power flow equations and the "a" and "d" factors introduced in Chapter 11.

For this reason, the value represented by a sensitivty coefficient is only good for small adjustments and the sensitivities must be recalculated often.

13.3.1 Sensitivity Coefficients of an AC Network Model

The following procedure is used to linearize the AC transmission system model for a power system. To start, we shall define two general equations giving the power injection at a bus. That is, the net power flowing into a transmission system from the bus. This function represents the power flowing into transmission lines and shunts at the bus:

$$\begin{aligned}
P_i(|\mathbf{E}|, \boldsymbol{\theta}) &= Re\left[\left(\sum_j E_i[(E_i - t_{ij}E_j)y_{ij}]^*\right) + E_i\left(E_i \sum_i y_{\text{shunt}_i}\right)^*\right] \\
Q_i(|\mathbf{E}|, \boldsymbol{\theta}) &= Im\left[\left(\sum_j E_i[(E_i - t_{ij}E_j)y_{ij}]^*\right) + E_i\left(E_i \sum_i y_{\text{shunt}_i}\right)^*\right]
\end{aligned} \tag{13.67}$$

where

$E_i = |E_i| \angle \theta_i$

t_{ij} = the transformer tap in branch ij

y_{ij} = the branch admittance

y_{shunt_i} = the sum of the branch and bus shunt admittances at bus i

Then, at each bus:

$$\begin{aligned}
P_i(|\mathbf{E}|, \boldsymbol{\theta}) &= P_{\text{gen}_i} - P_{\text{load}_i} \\
Q_i(|\mathbf{E}|, \boldsymbol{\theta}) &= Q_{\text{gen}_i} - Q_{\text{load}_i}
\end{aligned} \tag{13.68}$$

The set of equations that represents the first-order approximation of the AC network around the initial point is the same as generally used in the Newton power flow algorithm. That is:

$$\begin{aligned}
\sum \frac{\partial P_i}{\partial |E_j|} \Delta |E_j| + \sum \frac{\partial P_i}{\partial \theta_j} \Delta \theta_j + \sum \frac{\partial P_i}{\partial j_{ij}} \Delta t_{ij} &= \Delta P_{\text{gen}_i} \\
\sum \frac{\partial Q_i}{\partial |E_j|} \Delta |E_j| + \sum \frac{\partial Q_i}{\partial \theta_j} \Delta \theta_j + \sum \frac{\partial Q_i}{\partial t_{ij}} \Delta t_{ij} &= \Delta Q_{\text{gen}_i}
\end{aligned} \tag{13.69}$$

This can be placed in matrix form for easier manipulation:

$$\begin{bmatrix}
\dfrac{\partial P_1}{\partial |E_1|} & \dfrac{\partial P_1}{\partial \theta_1} & \cdots \\
\dfrac{\partial Q_1}{\partial |E_1|} & \dfrac{\partial Q_1}{\partial \theta_1} & \cdots \\
 & \vdots &
\end{bmatrix}
\begin{bmatrix} \Delta |E_1| \\ \Delta \theta_1 \\ \vdots \end{bmatrix}
=
\begin{bmatrix}
-\dfrac{\partial}{\partial t_{ij}}(P_i) & 1 & 0 \\
-\dfrac{\partial}{\partial t_{ij}}(Q_i) & 0 & 1
\end{bmatrix}
\begin{bmatrix} \Delta t_{ij} \\ \Delta P_{\text{gen}_i} \\ \Delta Q_{\text{gen}_i} \end{bmatrix} \tag{13.70}$$

This equation will be placed into a more compact format that uses the vectors $\mathbf{x}$ and $\mathbf{u}$, where $\mathbf{x}$ is the state vector of voltages and phase angles, and $\mathbf{u}$ is the vector of control variables. The control variables are the generator MW, transformer taps, and generator voltage magnitudes (or generator MVAR). Note that at any given generator bus we can control a voltage magnitude only within the limits of the unit VAR capacity. Therefore, there are times when the role of the state and control are reversed. Note that other controls can easily be added to this formulation. The compact form of Eq. 12.30 then is written:

$$[J_{px}]\Delta\mathbf{x} = [J_{pu}]\Delta\mathbf{u} \tag{13.71}$$

Now, we will assume that there are several transmission system dependent variables, $\mathbf{h}$, that represent, for example, MVA flows, load bus voltages, line amperes, etc., and we wish to find their sensitivity with respect to changes in the control variables. Each of these quantities can be expressed as a function of the state and control variables; that is, for example:

$$\mathbf{h} = \begin{bmatrix} \text{MVA flow}_{nm}(|\mathbf{E}|, \boldsymbol{\theta}) \\ |E_k| \end{bmatrix} \tag{13.72}$$

where $|E_k|$ represents only load bus voltage magnitude.

As before, we can write a linear version of these variables around the operating point

$$\Delta\mathbf{h} = \begin{bmatrix} \dfrac{\partial h_1}{\partial|E_1|} & \dfrac{\partial h_1}{\partial\theta_1} & \cdots \\ \dfrac{\partial h_2}{\partial|E_1|} & \dfrac{\partial h_2}{\partial\theta_1} & \cdots \\ & \vdots & \cdots \end{bmatrix} \begin{bmatrix} \Delta|E_1| \\ \Delta\theta_1 \\ \vdots \end{bmatrix} + \begin{bmatrix} \dfrac{\partial h_1}{\partial u_1} & \dfrac{\partial h_1}{\partial u_2} & \cdots \\ \dfrac{\partial h_2}{\partial u_1} & \dfrac{\partial u_2}{\partial h_2} & \cdots \\ & \vdots & \cdots \end{bmatrix} \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \\ \vdots \end{bmatrix} \tag{13.73}$$

where

h_1 = the line nm MVA flow

h_2 = the bus k voltage magnitude

Again, we can put this into a compact format using the vectors $\mathbf{x}$ and $\mathbf{u}$ as before:

$$\Delta\mathbf{h} = [J_{hx}]\Delta\mathbf{x} + [J_{hu}]\Delta\mathbf{u} \tag{13.74}$$

We will now eliminate the $\Delta\mathbf{x}$ variables; that is:

$$\Delta\mathbf{x} = [J_{px}]^{-1}[J_{pu}]\Delta\mathbf{u} \tag{13.75}$$

Then, substituting:

$$\Delta\mathbf{h} = [J_{hx}][J_{px}]^{-1}[J_{pu}]\Delta\mathbf{u} + [J_{hu}]\Delta\mathbf{u} \tag{13.76}$$

This last equation gives the linear sensitivity coefficients between the transmission system quantities, **h**, and the control variables, **u**.

13.4 LINEAR PROGRAMMING METHODS

The gradient and Newton methods of solving an OPF suffer from the difficulty in handling inequality constraints. Linear programming, however, is very adept at handling inequality constraints, as long as the problem to be solved is such that it can be linearized without loss of accuracy.

Figure 13.4 shows the type of strategy used to create an OPF using linear programming. The power flow equations could be for the DC representation, the decoupled set of AC equations, or the full AC power flow equations. The choice will affect the difficulty of obtaining the linearized sensitivity coefficients and the convergence test used.

In the formulation below, we show how the OPF can be structured as an LP. First, we tackle the problem of expressing the nonlinear input–output or cost functions as a set of linear functions. This is similar to the treatment in Section 7.9 for hydro-units. Let the cost function be $F_i(P_i)$ as shown in Figure 13.5.

We can approximate this nonlinear function as a series of straight-line segments as shown in Figure 13.6. The three segments shown will be represented as P_{i1}, P_{i2}, P_{i3}, and each segment will have a slope designated:

$$s_{i1},\ s_{i2},\ s_{i3}$$

then the cost function itself is

$$F_i(P_i) = F_i(P_i^{\min}) + s_{i1}P_{i1} + s_{i2}P_{i2} + s_{i3}P_{i3} \tag{13.77}$$

and

$$0 \le P_{ik} \le P_{ik}^{+} \quad \text{for } k = 1, 2, 3 \tag{13.78}$$

and finally

$$P_i = P_i^{\min} + P_{i1} + P_{i2} + P_{i3} \tag{13.79}$$

The cost function is now made up of a linear expression in the P_{ik} values.

In the formulation of the OPF using linear programming, we only have the control variables in the problem. We do not attempt to place the state variables into the LP, nor all the power flow equations. Rather, constraints are set up in the LP that reflect the influence of changes in the control variables only. In the examples we present here, the control variables will be limited to generator real power, generator voltage magnitude, and transformer taps. The control variables will be designated as the u variables (see earlier in this chapter).

The next constraint to consider in an LPOPF are the constraints that

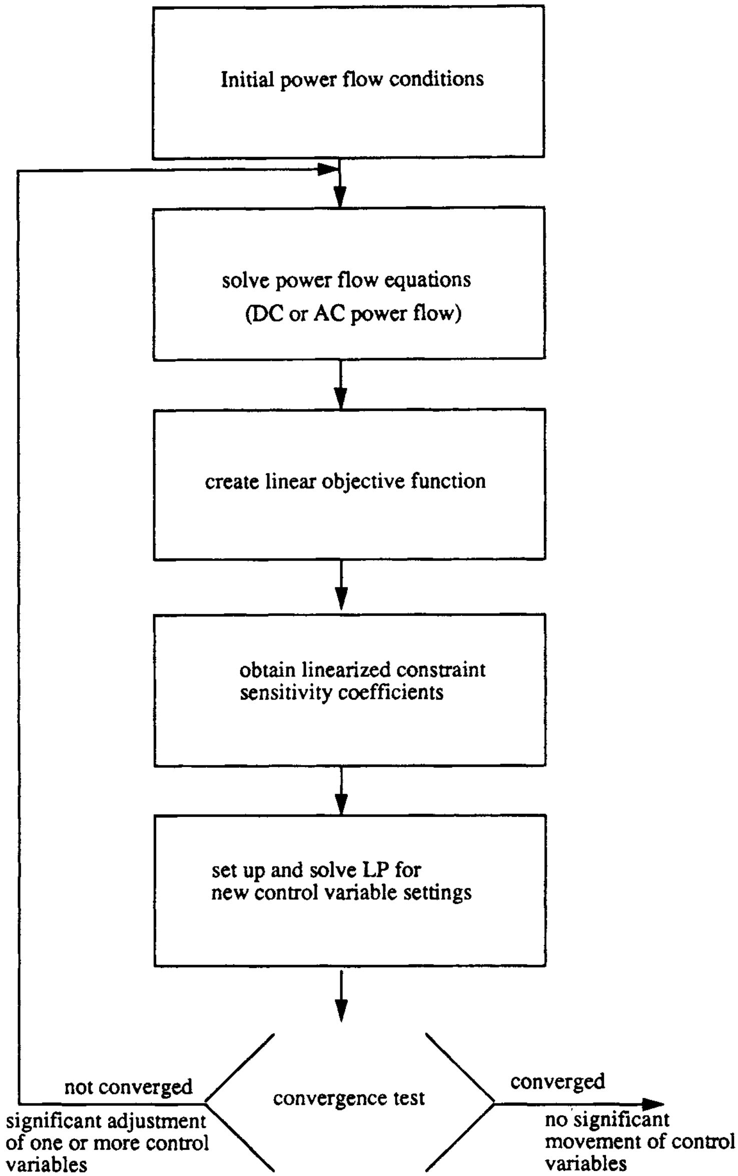

FIG. 13.4 Strategy for solution of the LPOPF.

represent the power balance between real and reactive power generated, and that consumed in the loads and losses. The real power balance equation is:

$$P_{\text{gen}} - P_{\text{load}} - P_{\text{loss}} = 0 \tag{13.80}$$

The loss term here represents the I^2R losses in the transmission lines and

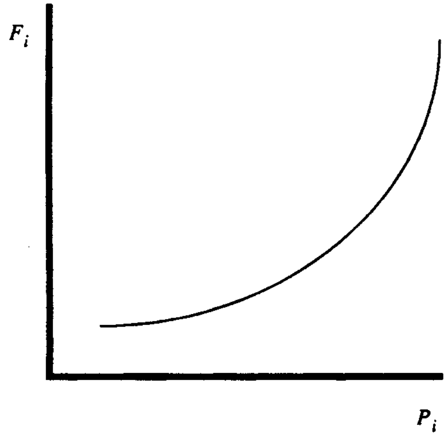

FIG. 13.5 A nonlinear cost function characteristic.

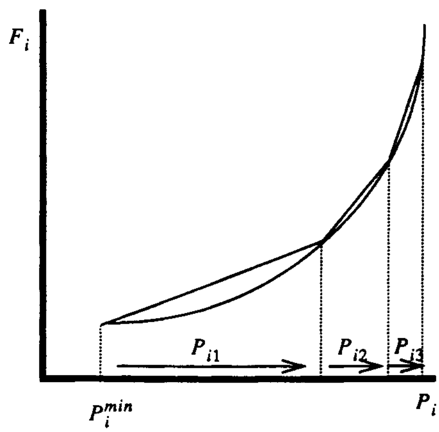

FIG. 13.6 A linearized cost function.

transformers. We can take derivatives with respect to the control variables, u, and this results in:

$$\sum_u \left(\frac{\partial P_{\text{gen}}}{\partial u}\right)\Delta u - \sum_u \left(\frac{\partial P_{\text{load}}}{\partial u}\right)\Delta u - \sum_u \left(\frac{\partial P_{\text{loss}}}{\partial u}\right)\Delta u = 0 \tag{13.81}$$

If we make the following substitution:

$$\Delta u = u - u^0 \tag{13.82}$$

then, the power balance equation becomes

$$\sum_u \left(\frac{\partial P_{\text{gen}}}{\partial u}\right)u - \sum_u \left(\frac{\partial P_{\text{load}}}{\partial u}\right)u - \sum_u \left(\frac{\partial P_{\text{loss}}}{\partial u}\right)u = K_p \tag{13.83}$$

where

$$K_p = \sum_u \frac{\partial P_{\text{gen}}}{\partial u} u^0 - \sum_u \frac{\partial P_{\text{load}}}{\partial u} u^0 - \sum_u \frac{\partial P_{\text{loss}}}{\partial u} u^0 \tag{13.84}$$

A similar equation can be written for the reactive power balance:

$$\sum_u \left(\frac{\partial Q_{\text{gen}}}{\partial u}\right)\Delta u - \sum_u \left(\frac{\partial Q_{\text{load}}}{\partial u}\right)\Delta u - \sum_u \left(\frac{\partial Q_{\text{loss}}}{\partial u}\right)\Delta u = 0 \tag{13.85}$$

where the loss term is understood to include I^2X as well as the charging from line capacitors and shunt reactors. A substitution using $\Delta u = u - u^0$, as above, can also be done here.

The LP formulation, so far, would need to restrict control variables to move only within their respective limits, but it does not yet constrain the OPF to optimize cost within the limits of transmission flows and load bus voltages. To add the latter type constraints, we must add a new constraint to the LP. For example, say we wish to constrain the MVA flow on line *nm* to fall within an upper limit:

$$\text{MVA flow}_{nm} \le \text{MVA flow}_{nm}^{\max} \tag{13.86}$$

We model this constraint by forming a Taylor's series expansion of this flow and only retaining the linear terms:

$$\text{MVA flow}_{nm} = \text{MVA flow}_{nm}^0 + \sum_u \left(\frac{\partial}{\partial u} \text{MVA flow}_{nm}\right)\Delta u \le \text{MVA flow}_{nm}^{\max} \tag{13.87}$$

Again, we can substitute $\Delta u = u - u^0$ so we get:

$$\sum_u \left(\frac{\partial}{\partial u} \text{MVA flow}_{nm}\right) u \le \text{MVA flow}_{nm}^{\max} - K_f \tag{13.88}$$

where

$$K_f = \text{MVA flow}_{nm}^0 + \sum_u \frac{\partial}{\partial u} \text{MVA flow}_{nm} u^0 \tag{13.89}$$

Other constraints such as voltage magnitude limits, branch MW limits, etc., can be added in a similar manner. We add as many constraints as necessary to constrain the power system to remain within its prescribed limits. Note, of course, that the derivatives of P_{loss} and MVA flow_{nm} are obtained from the linear sensitivity coefficient calculations presented in the previous section.

13.4.1 Linear Programming Method with Only Real Power Variables

As an introduction to the LPOPF, we will set up and solve a power system example which only has generator real powers as control variables. Further, the model for the power system power balance constraint will assume that load is constant and that the losses are constant. Finally, since the entire model used in the LP is based on a MW-only formulation, we shall use the "a" and "d" factors derived in Chapter 11 to model the effect of changes in controls on the constraints. As indicated in Figure 13.4, we shall solve the LP and then make the adjustments to the control variables and solve a power flow in each main iteration. This guarantees that the total generation equals load plus losses, and that the MW flows are updated properly. The cost functions can be treated as before using multiple segmented "piecewise linear" approximations.

The "power balance" equation for this case is as follows:

$$P_1 + P_2 + \ldots + P_{\text{ref}} = P_{\text{load}} + P_{\text{losses}} = \text{constant} \tag{13.90}$$

To constrain the power system, we need the expansion of the constraints, such as MW flows, bus voltages, etc., as linear functions of the control variables. In this case, the linear control variables will be represented as a vector $\mathbf{u}$:

$$u = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{\text{ref}} \end{bmatrix} \tag{13.91}$$

This is done with the linear sensitivity approach, as derived in the previous section. The result is a set of constraints:

$$\mathbf{h}(\mathbf{u}) \le \mathbf{h}^+ \tag{13.92}$$

which is written as

$$\mathbf{h}(\mathbf{u}) = h(\mathbf{u}^0) + \frac{\partial h}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u}^0) \le \mathbf{h}^+ \tag{13.93}$$

However, we shall observe that the derivatives $\partial h/\partial \mathbf{u}$ can be replaced with the "a" sensitivity coefficients developed in Chapter 11.

Thus, for a MW flow constraint on line rs we have:

$$\text{MW}_{rs} = \text{MW}_{rs}^0 + \sum_u a_{rs-u}(u - u^0) \le \text{MW}_{rs}^{\max} \tag{13.94}$$

or

$$\text{MW}_{rs} = \sum_u a_{rs-u} u \le \text{MW}_{rs}^{\max} - \left(\text{MW}_{rs}^0 - \sum_u a_{rs-u} u^0\right) \tag{13.95}$$

TABLE 13.1 Line Flows: Power Flow 0

Line	Limit	MW Flow
1-2	30	28.69
1-4	50	43.58
1-5	40	35.60
2-3	20	2.93
2-4	40	33.09
2-5	20	15.51
2-6	30	26.25
3-5	20	19.12
3-6	60	43.77
4-5	20	4.08
5-6	20	1.61

Similar constraints are added for any power system network quantity that is to held within its limit.

EXAMPLE 13C

We shall use the LPOPF reduced model method to solve an OPF problem. An LP and an AC power flow will be used to solve a series of dispatch problems. The transmission system will be the six bus system introduced in Chapter 4, the MW limits on the transmission lines will be those introduced in Example 11B and shown in Table 13.1. The generator cost functions are those found in Example 4E and linearized as shown below.

We shall solve a series of LP–AC power flow calculations as follows.

Step 0

Run a base AC power flow (this will be the AC power flow shown in Figure 4.8 and it will be designated as POWER Flow 0 in numbering the various power flow calculations in this example). Looking at Figure 4.8 and the limit set we are using from Example 11B, also shown below, we note that there are no overloads.

The generation values for this power flow are:

$$P_1 = 107.87 \text{ MW}, P_2 = 50 \text{ MW}, \quad \text{and} \quad P_3 = 69 \text{ MW} \quad \textit{power flow } 0\text{: } \textit{result}$$

The total cost for this initial dispatch is 3189.4 ₽/h.

Step 1

We now set up the LP to solve for the optimum cost with only the power balance equation in the LP constraint set. By the nature of the cost curve

TABLE 13.2 Generator Unit Break Point MWs

Unit	Break Point 1 (unit min)	Break Point 2	Break Point 3	Break Point 4 (unit max)
1	50	100	160	200
2	37.5	70	130	150
3	45	90	140	180

TABLE 13.3 Generator Cost Curve Segment Slope

Generator	s_{i1}	s_{i2}	s_{i3}
1	12.4685	13.0548	13.5875
2	11.2887	12.1110	12.8222
3	11.8333	12.5373	13.2042

segments, we also incorporate the limits on the generators. The generator cost functions are as follows:

Generator on bus 1: $F_1(P_1) = 213.1 + 11.669P_1 + 0.00533P_1^2$ Ɍ/h
with limits of: $50.0 \text{ MW} \leq P_1 \leq 200.0 \text{ MW}$

Generator on bus 2: $F_2(P_2) = 200.0 + 10.333P_2 + 0.00889P_2^2$ Ɍh
with limits of: $37.5 \text{ MW} \leq P_2 \leq 150.0 \text{ MW}$

Generator on bus 3: $F_3(P_3) = 240.0 + 10.833P_3 + 0.00741P_3^2$ Ɍ/h
with limits of: $45.0 \text{ MW} \leq P_3 \leq 180.0 \text{ MW}$

The LP will be run with the unit cost functions broken into three straight-line segments such that the break points are located as shown in Table 13.2. The generator cost function segment slopes are computed as follows:

$$s_{if} = \frac{F_i(P_{ij}^+) - F_i(P_{ij}^-)}{P_{ij}^+ - P_{ij}^-} \tag{13.96}$$

where P_{ij}^+ and P_{ij}^- are the values of P_i at the end of the j^{th} cost curve segment. The values are shown in Table 13.3. The segment limits are shown in Table 13.4.

The LP cost function is:

$$\begin{aligned}&[F_1(P_1^{\min}) + 12.4685P_{11} + 13.0548P_{12} + 13.5878P_{13}]\\&\quad + [F_2(P_2^{\min}) + 11.2887P_{21} + 12.1110P_{22} + 12.8222P_{23}]\\&\quad + [F_3(P_3^{\min}) + 11.8333P_{31} + 12.5373P_{32} + 13.2042P_{33}]\end{aligned} \tag{13.97}$$

TABLE 13.4 Segment Limits

Segment	Min MW	Max MW
P_{11}	0	50
P_{12}	0	60
P_{13}	0	40
P_{21}	0	32.5
P_{22}	0	60
P_{23}	0	20
P_{31}	0	45
P_{32}	0	50
P_{33}	0	40

Since the $F_i(P_i^{\min})$ terms are constant, we can drop them in the LP. Then, the cost function becomes:

$$\begin{aligned}&12.4685P_{11} + 13.0548P_{12} + 13.5878P_{13} + 11.2887P_{21}\\ &\quad + 12.1110P_{22} + 12.8222P_{23} + 11.8333P_{31}\\ &\quad + 12.5373P_{32} + 13.2042P_{33}\end{aligned} \tag{13.98}$$

The generation, load, and losses equality constraint is

$$P_1 + P_2 + P_3 = P_{\text{load}} + P_{\text{losses}} \tag{13.99}$$

The load is 210 MW and the losses from the initial power flow are 7.87 MW. Substituting the equivalent expression for each generator's output in terms of its three linear segments, we obtain:

$$\begin{aligned}&P_1^{\min} + P_{11} + P_{12} + P_{13} + P_2^{\min} + P_{21} + P_{22} + P_{23} + P_3^{\min}\\ &\quad + P_{31} + P_{32} + P_{33} = P_{\text{load}} + P_{\text{losses}}\end{aligned} \tag{13.100}$$

This results in the following after the $P_i^{\min}$, P_{load}, and P_{loss} values are substituted:

$$\begin{aligned}&P_{11} + P_{12} + P_{13} + P_{21} + P_{23} + P_{33} + P_{31} + P_{32} + P_{33}\\ &\quad = 210 + 7.87 - 50 - 37.5 - 45 = 85.37\end{aligned} \tag{13.101}$$

We now solve the LP with the cost function and equality constraint given above, and with the six variables representing the generator outputs. The solution to this LP is shown in Table 13.5.

TABLE 13.5 First LP Solution

Variable	Min MW	Solution MW	Max MW
P_{11}	0	0.0	50
P_{12}	0	0.0	60
P_{13}	0	0.0	40
P_{21}	0	32.5	32.5
P_{22}	0	7.87	60
P_{23}	0	0.0	20
P_{31}	0	45.0	45
P_{32}	0	0.0	50
P_{33}	0	0.0	40

The total generation on each generator is:

$$P_i = P_i^{\min} + P_{i1} + P_{i2} + P_{i3} \tag{13.102}$$

then the generator optimal outputs are

$$P_1 = 50 \text{ MW}, P_2 = 77.87 \text{ MW}, \quad \text{and} \quad P_3 = 90 \text{ MW} \; \textit{LP 1: result}$$

Note that this solution of necessity will have only one of the variables not at a break point while the others will be at a break point. Note also that the output on bus 1 is at its low limit. When we substitute these values for the generation at buses 1, 2, and 3, and run the power flow, we get the following:

$$P_1 = 48.83 \text{ MW}, P_2 = 77.87 \text{ MW}, \quad \text{and} \quad P_3 = 90 \text{ MW} \; \textit{power flow 1: result}$$

The total cost for this dispatch is 3129.1 ℞/h. This illustrates the fact that the LP uses a linear model of the power system and when we put its results into a nonlinear model, such as the power flow, there are bound to be differences. Since the losses have changed (to 6.70 MW), the power output of the reference bus must decrease to balance the power flow. However, the solution to the optimal LPOPF has the reference-bus power output below its minimum of 50 MW. To correct this condition we set up another LP solution with the same cost function but with a slightly different equality constraint that reflects the new value of losses. The result of this LP is:

$$P_1 = 50 \text{ MW}, P_2 = 76.7 \text{ MW}, \quad \text{and} \quad P_3 = 90 \text{ MW} \; \textit{LP 1.1: result}$$

Once again, we enter these results into the power flow and obtain:

$$P_1 = 49.99 \text{ MW}, P_2 = 76.7 \text{ MW} \quad \text{and} \quad P_3 = 90 \text{ MW} \; \textit{power flow 1.1 result}$$

TABLE 13.6 Line Flows: Power Flow 1.1

Line	Limit	MW Flow
1-2	30	4.28
1-4	50	25.60
1-5	40	20.11
2-3	20	−6.42
2-4	40	48.75[a]
2-5	20	17.75
2-6	30	20.88
3-5	20	28.91[a]
3-6	60	54.63
4-5	20	1.84
5-6	20	3.87

[a] Overloaded line.

The total cost for this dispatch is 3129.6 ℞/h and the losses are 6.7 MW. This represents the least cost dispatch that we shall obtain in this example. As constraints are added later to meet the flow limits, the cost will increase.

Note also that we have two overloads on the optimum cost dispatch as shown in Table 13.6.

Step 2

The LP and power flow executions in step 1 resulted in a less-costly dispatch than the original power flow, but in doing so we have overloaded two transmission lines. We shall refer to these overloads as $(n - 0)$ overloads. This notation means that there are n lines minus zero outages in the network at the time of the overload. [Later we shall use the notation $(n - 1)$ to indicate that there are n lines minus one line (that is, a single-line outage) in the network at the time of the overloads. This notation can be used for further levels of overload such as $(n - 2)$, $(n - 3)$, etc. However, many electric utility transmission operations departments only go as far as $(n - 1)$ in dispatching their systems.]

We must redispatch the power system at this point to remove the $(n - 0)$ overloads. To do this, we add two constraints to the LP, one for each overloaded line. The power flow constraint on line 2-4 is modeled as:

$$f_{2\text{-}4} = f^0_{2\text{-}4} + a_{2\text{-}4,1}(P_1 - P^0_1) + a_{2\text{-}4,2}(P_2 - P^0_2) + a_{2\text{-}4,3}(P_3 - P^0_3) \leq 40 \tag{13.103}$$

Substituting 48.75 for $f^0_{2\text{-}4}$, 76.7 for P^0_2, and 90 for P^0_3, we get the following for the constraint for line 2-4 (note that $a_{2\text{-}4,1} = 0$) and, finally, we expand P_2 and P_3 in terms of the segments:

$$48.75 + 0.31(37.5 + P_{21} + P_{22} + P_{23} - 76.7) + 0.22(45 + P_{31} + P_{32} + P_{33} - 90) \le 40 \qquad (13.104)$$

or

$$0.31P_{21} + 0.31P_{22} + 0.31P_{23} + 0.22P_{31} + 0.22P_{32} + 0.22P_{33} \le 13.302 \qquad (13.105)$$

The constraint for line 3-5 is built similarly and results in:

$$0.06P_{21} + 0.06P_{22} + 0.06P_{23} + 0.29P_{31} + 0.29P_{32} + 0.29P_{33} \le 6.492 \qquad (13.106)$$

The solution to the LP gives:

$P_1 = 87.02$ MW, $P_2 = 70.0$ MW and $P_3 = 59.66$ MW *LP 2: result*

Also note that only the first transmission line constraint is binding in the LP, the remaining constraint is "slack," that is, it is not being forced up against its limit. When these values are put into the power flow we obtain:

$P_1 = 87.54$ MW, $P_2 = 70.0$ MW and $P_3 = 59.66$ MW *power flow 2: result*

The flows on the two constrained lines are:

$f_{2\text{-}4} = 39.40$ MW and $f_{3\text{-}5} = 20.36$ MW

The total operating cost has now increased to 3155.0 Ɽ/h.

We now run another complete LP–power flow iteration to account for changes in losses and to bring the constraints closer to their limits. The solution to the second-iteration LP gives:

$P_1 = 86.16$ MW, $P_2 = 73.3$ MW and $P_3 = 57.73$ MW *LP 2.1: result*

Both transmission line constraints are binding in the second LP. When these values are put into the power flow we obtain:

$P_1 = 86.16$ MW, $P_2 = 73.3$ MW and $P_3 = 57.73$ MW *power flow 2.1: result*

The flows on the two constrained lines are:

$f_{2\text{-}4} = 39.99$ MW and $f_{3\text{-}5} = 20.06$ MW

The total operating cost has now decreased slightly to 3153.3 Ɽ/h. There are no more $(n - 0)$ line overloads.

TABLE 13.7 Line Flows: Power Flow 2.1 (with Line 2-3 Out)

Line	Limit	MW Flow
1-2	30	18.1
1-4	50	36.37
1-5	40	31.74
2-3	20	—
2-4	40	40.73[a]
2-5	20	19.19
2-6	30	31.11[a]
3-5	20	18.26
3-6	60	39.47
4-5	20	4.59
5-6	20	1.17

[a] Overloaded line.

Step 3

We have now achieved an optimal dispatch with all $(n - 0)$ overloads met. This dispatch will satisfy generation and all line flow limits; however, if we have a transmission line outage contingency, we may have overloads. By modeling the first contingency overloads, or the so-called $(n - 1)$ overloads, we can guarantee that should the contingency outage take place, there would be no resulting overloads. This is the scheme involved in security-constrained OPF, or SCOPF, and is the subject of Section 13.5.

In this example, to make matters simple we shall only study the result of one contingency outage. In our sample system, we shall start from the result of power flow 2.1 and take out line 2-3. The flows that result from this contingency power flow are shown in Table 13.7.

We now must form a new LP that has the generation, load, losses equality constraint and the original two $(n - 0)$ line flow constraints done in step 2, and two new constraints for each of the $(n - 1)$ overloads (i.e., on line 2-4 and line 2-6). To model line 2-4 with line 2-3 removed, we use the following constraint, as derived in Appendix 11A of Chapter 11.

$$\Delta f_l^k = \sum_i (a_{li} + d_{l,k} a_{ki}) \Delta P_i \tag{13.107}$$

This now becomes:

$$f_l^k = \sum_i (a_{li} + d_{l,k} a_{ki})(P_i - P_i^0) + f_l^0 \le f_l^{\max} \tag{13.108}$$

The new LP has five constraints. The first result of this LP gives:

$P_1 = 91.39$ MW, $P_2 = 66.96$ MW, and $P_3 = 58.84$ MW *LP 3: result*

The ($n = 0$) constraint on line 3-5 is binding and the ($n - 1$) constraint on line 2-6 is binding. When these values are put into the power flow, we obtain (note that this power flow has all lines in):

$P_1 = 91.52$ MW, $P_2 = 66.96$ MW, and $P_3 = 58.8$ MW *power flow 3: result*

The flows on the two ($n - 0$) constrained lines are:

$$f_{2\text{-}4} = 38.23 \text{ MW} \quad \text{and} \quad f_{3\text{-}5} = 19.94 \text{ MW}$$

A second power flow with line 2-3 out is also run with the same generation values. The results of this power flow show that the two ($n - 1$) flow constraints are:

$$f_{2\text{-}4}^{\text{contingency}} = 38.86 \text{ MW} \quad \text{and} \quad f_{2\text{-}6}^{\text{contingency}} = 30.00 \text{ MW}$$

The total operating cost has now increased to 3160.5 Ɍ/h. A complete second iteration of the LP and power flows is run and results in the following power flows:

$P_1 = 90.53$ MW, $P_2 = 67.92$ MW, and $P_3 = 58.84$ MW *power flow 3.1: result*

The flows on the two ($n - 0$) constrained lines are:

$$f_{2\text{-}4} = 38.54 \text{ MW} \quad \text{and} \quad f_{3\text{-}5} = 20.00 \text{ MW}$$

A second power flow with line 2-3 out is also run with the same generation values. The results of this power flow show that the two ($n - 1$) flow constraints are:

$$f_{2\text{-}4}^{\text{contingency}} = 39.18 \text{ MW} \quad \text{and} \quad f_{2\text{-}6}^{\text{contingency}} = 30.09 \text{ MW}$$

The total operating cost has now increased to 3159.1 Ɍ/h.

13.4.2 Linear Programming with AC Power Flow Variables and Detailed Cost Functions

OPF programs that optimize the AC power flow of a power system go beyond the LPOPF introduced in the last section, in several respects.

First, they do not usually use fixed break points. Rather, the break points are added as needed as the solution progresses and can become close enough so that no error is perceptible between the piecewise linear approximation and the true nonlinear input–output curve of the generators. "Second, the AC quantities of voltage magnitude and perhaps phase angle become variables in the LP and the constraints are set up as linear functions using the sensitivity coefficients methods shown in Section 13.3. Usually, however, the nonlinear representations of the bus power and reactive power injections and the line or transformer MVA flows are not well represented as linear functions. To cope with the nonlinear nature of these constraints involves restricting the movement of each variable and then relinearlizing the equality and inequality constraints quite often. The result is an LP that "converges" on the optimal AC power flow, meeting all the power flow equality constraints and inequality constraints.

Reference 9 is an example of such an OPF code built around an LP.

13.5 SECURITY-CONSTRAINED OPTIMAL POWER FLOW

In Chapter 11, we introduced the concept of security analysis and the idea that a power system could be constrained to operate in a secure manner. Programs which can make control adjustments to the base or pre-contingency operation to prevent violations in the post-contingency conditions are called "security-constrained optimal power flows," or SCOPF.

We have seen previously that an OPF is distinguished from an economic dispatch by the fact that it constantly updates a power flow of the transmission system as it progresses toward the minimum of the objective function. One advantage of having the power flow updated is the fact that constraints can be added to the OPF that reflect the limits which must be respected in the transmission system. Thus, the OPF allows us to reach an optimum with limits on network components recognized.

An extension to this procedure is to add constraints that model the limits on components during contingency conditions. That is, these new "security constraints" or "contingency constraints" allow the OPF to meet precontingency limits as well as post-contingency limits. There is a price to pay, however, and that is the fact as we iterate the OPF with an AC power flow, we must also run power flows for all the contingency cases being observed. This is illustrated in Figure 13.7.

The SCOPF shown in Figure 13.7 starts by solving an OPF with $(n - 0)$ constraints only. Only when it has solved for the optimal, constrained conditions is the contingency analysis executed. In Figure 13.7, the contingency analysis starts by screening the power system and identifying the potential worst-contingency cases. As was pointed out in Chapter 11, not all of these cases are going to result in a post-contingency violation and it is important to limit the number of full power flows that are executed. This is especially important in the SCOPF, where each contingency power flow may result in new contingency constraints being added to the OPF. We assume here that only the M worst cases screened by the screening algorithm are added. It is possible to make $M = 1$, in which case only the worst potential contingency is added.

Next, all the $(n - 1)$ contingency cases that are under consideration must be solved by running a power flow with that contingency reflected in alterations to the power flow model. When the power flow results in a security violation,

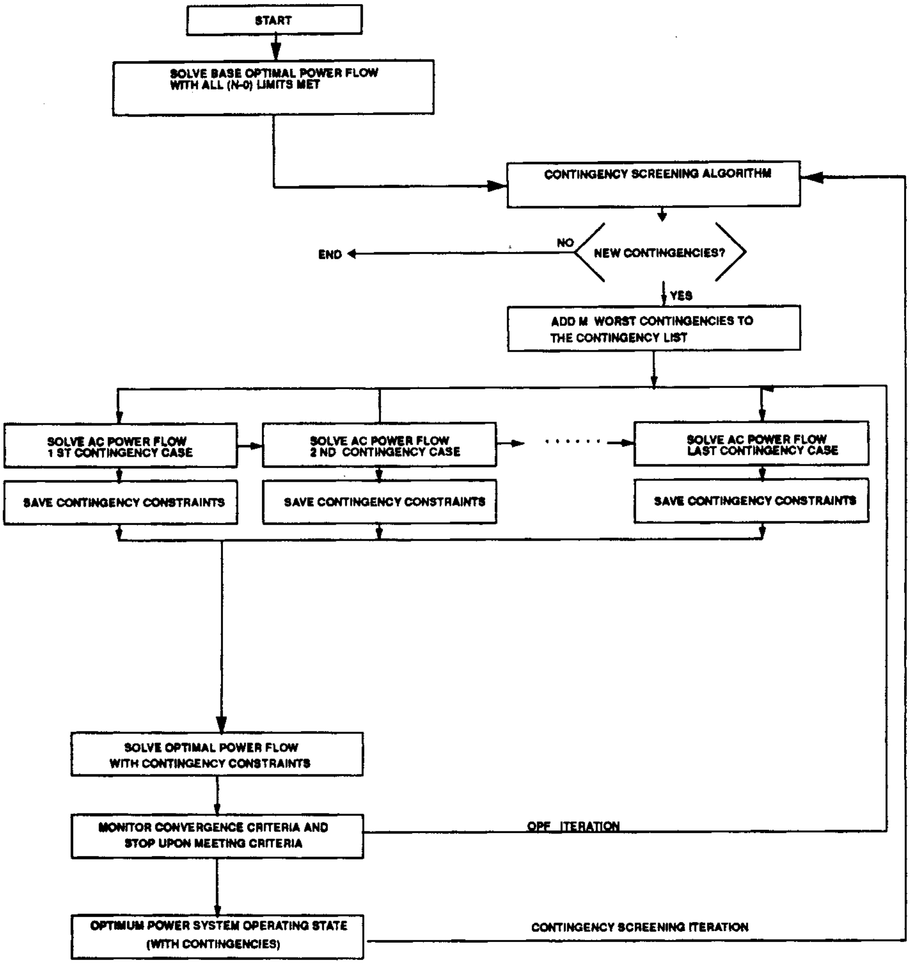

FIG. 13.7 Security-constrained optimal power flow.

the power system model is used to create a contingency constraint. In fact, what is done is to run a network sensitivity calculation (See Section 13.3) on the model with the contingency outage and save the resulting constraint sensitivities. When all contingency power flows are complete, all the contingency constraints are added to the OPF model and it is solved.

Note, in Figure 13.7, there are two main loops to be executed. The loop labeled "OPF Iteration" requires the OPF and each of the contingency power flows to be re-executed until the OPF has solved with all contingency constraints met. Next, the outer loop labeled "Contingency Screening Iteration" is tried. If the contingency screening algorithm does not pick up any new contingencies the SCOPF can end; if new contingencies are found, it must add them to the list and continue.

Why is all this necessary? The optimum operation conditions for a power system will often result in violation of system security. This is especially true when a large amount of interchange power is available at a favorable price. In this instance, the selling power system can be modeled in the OPF with its price of production set accordingly, and the OPF will then raise the interchange up to the point where transmission system components are limiting. Now, when the contingency analysis is run, there may be many cases which result in contingency violations and the OPF, with contingency constraints added, will have to back off the interchange power in order to meet the contingency limits.

It should also be noted that when some contingency constraints are added to the OPF, it will redispatch generation, and adjust voltages and transformers to meet these constraints. The process of adjustments may result in many new contingency violations when the screening algorithm and the power flows are run. The need to iterate between the OPF and the contingency screening represents an effort to find the "most constraining" contingencies.

SCOPF was introduced as step 3 in Example 13C and will also be illustrated in Example 13D, which follows.

EXAMPLE 13D

This example shows the results of running the same six-bus case used in Example 13C, with the same generator cost functions. However, we now are using a full AC OPF so that we will use line MVA limits and bus voltage limits as well. The MVA limits are shown in Table 13.8. The bus voltages are also limited, with bus 5 being the only one to hit its upper limit of 1.0 pu voltage magnitude.

The full AC OPF has six control variables: three generator outputs and three generator voltage magnitude schedules. In addition, the AC OPF can be used to minimize either MW losses, or to minimize operating cost. Table 13.9 summarizes these results.

TABLE 13.8 Line MVA Flows: Power Flow 0

Line	MVA Limit	MVA Flow
1-2	40	32.57
1-4	60	48
1-5	40	37.34
2-3	40	12.61
2-4	60	56.71
2-5	30	21.83
2-6	90	29.03
3-5	70	30.04
3-6	80	74.86
4-5	20	6.41
5-6	40	9.80

TABLE 13.9 Full AC OPF Results

Case	P_1	P_2	P_3	$\|E_1^{sched}\|$	$\|E_2^{sched}\|$	$\|E_3^{sched}\|$	MW Losses	Cost	Binding Constraints
Base case	107.9	50.0	60.0	1.05	1.0499	1.0429	7.87	3189.4	Generators 2 and 3 at max VAR limit
Min cost, adjust generator MW only	86.9	59.3	71.0	1.05	1.05	1.0458	7.14	3157.9	Line 2-4 MVA limit, generator 3 at max VAR limit
Min losses, adjust generator voltage only	107.1	50.0	60.0	1.05	1.0429	1.0499	7.1	3179.5	Bus 5 max voltage, line 1-5 MVA limit, generator 3 at max VAR limit
Min cost with generator MW adjustment, then min losses with adjustment of gen voltages	86.3	59.3	71.0	1.05	1.0429	1.0499	6.54	3150.3	Bus 5 max voltage, generator 3 at max VAR limit
Min cost with both generator MW and voltage adjustment	52.0	87.5	77.0	1.05	1.0429	1.0499	6.41	3124.6	Bus 5 max voltage, line 2-4 MVA limit, generator 3 at max VAR limit
Min cost, adjust generator MW with line 3-6 out	70.0	91.8	62.0	1.05	1.031	1.07	15.05	3219.7	Before, OPF line 1-5 has 12% MVA overload; after, line 2-4 at MVA limit, generator at max VAR limit

Note the variety of ways that a power system can be optimized using an OPF. For example, some power system operators may wish to simply reduce system losses through the adjustment of generator voltage schedules—this is often done with hydrosystems where the generator MW outputs must be kept on a fixed time schedule to meet hydro-requirements.

13.6 INTERIOR POINT ALGORITHM

In 1984, Karmarkar (refeence 10) presented a new solution algorithm for linear programming problems that did not solve for the optimal solution by following a series of points that were on the "constraint boundary" but, rather, followed a path through the interior of the constraints directly toward the optimal solution on the constraint boundary. This solution was much faster than conventional LP algorithms.

In 1986, Gill *et al.* (reference 11) showed the relationship between Karmarkar's algorithm and the so-called "logarithmic barrier function algorithm." This algorithm has become the basis for many OPF solution algorithms and is explained in reference 12.

In this derivation, no distinction is made between the control variables and the state variables; rather, all variables are considered in the $\mathbf{x}$ vector. The objective function will be $f(\mathbf{x})$. The constraints will be brokem into equality constraints and inequality constraints. The equality constraints are $\mathbf{g}(\mathbf{x}) = 0$ and the inequality constraints are

$$\mathbf{h}^{-} \leq h(\mathbf{x}) \leq \mathbf{h}^{+} \tag{13.109}$$

where the $\mathbf{h}^{-}$ and $\mathbf{h}^{+}$ vectors are the lower and upper limits on the inequality constraints, respectively. Finally, we restrict the variables themselves to be within lower and upper bounds

$$\mathbf{x}^{-} \leq \mathbf{x} \leq \mathbf{x}^{+} \tag{13.110}$$

The first step in transforming this problem is to add slack variables so that all the equations become equality constraints. We then obtain the following set of equations:

$$\begin{gathered} \min f(\mathbf{x}) \\ g(\mathbf{x}) = 0 \\ h(\mathbf{x}) + \mathbf{s}_h = \mathbf{h}^{+} \\ \mathbf{s}_h + \mathbf{s}_{sh} = \mathbf{h}^{+} - \mathbf{h}^{-} \\ \mathbf{x} + \mathbf{s}_x = \mathbf{x}^{+} \\ \mathbf{x} - \mathbf{x}^{-} \geq 0,\ \mathbf{s}_x,\ \mathbf{s}_h,\ \mathbf{s}_{sh} \geq 0 \end{gathered} \tag{13.111}$$

Note that we now have a set of equations with all equality constraints except the final consisting of nonnegativity conditions on $\mathbf{x} - \mathbf{x}^-$ and the slack variables. These nonnegativity conditions are handled by adding what is called a "logarithmic barrier function" to the objective. Basically, this is a form of penalty function which becomes very large as the function or variable gets close to zero. The new objective function then looks like:

$$f_\mu = f(\mathbf{x}) - \mu \sum_j \ln(x - x^-)_j - \mu \sum_j \ln(\mathbf{s}_x)_j - \mu \sum_i \ln(\mathbf{s}_h)_i - \mu \sum_i \ln(\mathbf{s}_{sh})_i \tag{13.112}$$

The parameter, μ, is called the "barrier parameter" and is a positive number that is forced to go to zero as the algorithm converges to the optimum. This then presents us with the Lagrange equation:

$$\begin{aligned} \mathscr{L}_\mu = {} & f(\mathbf{x}) - \boldsymbol{\lambda}^T g(\mathbf{x}) - \boldsymbol{\lambda}_h^T[\mathbf{h}^+ - \mathbf{s}_h - h(\mathbf{x})] \\ & - \boldsymbol{\lambda}_{sh}^T(\mathbf{h}^+ - \mathbf{h}^- - \mathbf{s}_h - \mathbf{s}_{sh}) - \boldsymbol{\lambda}_x^T(\mathbf{x}^+ - \mathbf{x} - \mathbf{s}_x) \\ & - \mu \sum_j \ln(x - x^-)_j - \mu \sum_j \ln(\mathbf{s}_x)_j - \mu \sum_i \ln(\mathbf{s}_h)_i - \mu \sum_i \ln(\mathbf{s}_{sh})_i \end{aligned} \tag{13.113}$$

The solution to this Lagrangian equation is obtained by setting its gradient to zero:

$$\begin{aligned} \nabla_x \mathscr{L}_\mu &= \nabla f(\mathbf{x}) - \nabla_g(\mathbf{x})^T \boldsymbol{\lambda} + \nabla h(\mathbf{x})^T \boldsymbol{\lambda}_h + \boldsymbol{\lambda}_x - \mu(\mathbf{x} - \mathbf{x}^-)^{-1} e = 0 \\ \nabla_{s_h} \mathscr{L}_\mu &= \boldsymbol{\lambda}_h + \boldsymbol{\lambda}_{sh} - \mu_h^{-1} e = 0 \\ \nabla_{sh} \mathscr{L}_\mu &= \boldsymbol{\lambda}_{sh} - \mu s_{sh}^{-1} e = 0 \\ \nabla_{s_x} \mathscr{L}_\mu &= \boldsymbol{\lambda}_x - \mu s_s^{-1} e = 0 \\ \nabla_\lambda \mathscr{L}_\mu &= -g(\mathbf{x}) \\ \nabla_{\lambda_{sh}} \mathscr{L}_\mu &= h(\mathbf{x}) + s_h - h^+ \\ \nabla_{\lambda_x} \mathscr{L}_\mu &= \mathbf{x} + s_x - x^+ \\ \nabla_{\lambda_h} \mathscr{L}_\mu &= s_h + s_{sh} - h^+ + h^- \end{aligned} \tag{13.114}$$

These nonlinear equations are then solved iteratively by Newton's method, and the value of μ is adjusted toward zero.

The solution produces the values of the dual variables, some of which are the marginal costs for the real and reactive power at the buses. These bus incremental costs, BICs are the subject of the next section. Note that in Chapter 10, the BICs were calculated using an interior point OPF.

13.7 BUS INCREMENTAL COSTS

If we take the classical Lagrange equation for an optimal power flow:

$$\mathscr{L}(\mathbf{x}, \mathbf{u}, \mathbf{p}) = \mathrm{f}(\mathbf{x}, \mathbf{u}) + \lambda^t \mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{p}) \tag{13.115}$$

and we asume that we have an optimal solution to this equation, then we can ask an interesting question: "What is the change in the optimal operating cost if we change one of the parameters **p**?" More specifically: 'What is the change in optimal operating cost if we change the power produced or consumed at a bus in the network?" Thus, what we want is the following derivative:

$$\frac{\partial \mathscr{L}}{\partial P_i}$$

If we expand the Lagrange equation as follows:

$$\mathscr{L}(\mathbf{x}, \mathbf{u}, \mathbf{p}) = \sum_{\text{gen}} F_i(P_i) + F_{\text{ref}}[P_{\text{ref}}(/E|, \theta)] + [\lambda_1 \lambda_2, \ldots, \lambda_m] \begin{bmatrix} P_i^{\text{net}} - P_i(|\mathbf{E}|, \theta) \\ Q_i^{\text{net}} - Q_i(|\mathbf{E}|, \theta) \\ \vdots \end{bmatrix} \tag{13.116}$$

The derivative of $\mathscr{L}$ with respect to P_i is simple, since the parameters only appear in the second part of the Lagrange equation. The resulting derivative for bus i is:

$$\frac{\partial \mathscr{L}}{\partial P_i} = \lambda_i \tag{13.117}$$

We see that the interpretation of the vector of Lagrange multipliers is that they indicate the increment in optimal cost with respect to small changes in the parameters of the network. In the case of small change in power, either consumed or produced at a bus, the Lagrange multiplier for that bus then indicates the incremental cost that will be incurred as a result of this change. This cost has been given the name "bus incremental cost" or BIC and is the same incremental cost we dealt with in the beginning of the text, where we derived the incremental cost of delivery of power from a generator. A power system is in economic dispatch when the BIC for each generator matches the generator's own incremental cost for the power it is producing.

The BIC is a useful concept for nondispatched generator buses and for evaluating the marginal cost of wheeling. In some proposed schemes, this bus incremental cost is used to establish the spot market price for energy.

One point is worth noting before we leave this topic. The above discussion assumed that one has the vector of Lagrange multipliers for an optimal solution. However, depending on the method used to solve the OPF, this may not be the case. Certainly, in the case of the OPF that is based on linear programming, the λ values are not available unless a special formulation is used—yet we need the BICs for the buses.

The Lagrange equation at the optimal solution can be used to solve for the Lambda vector, even through it was not used in the OPF alogorithm. This is because, at the optimal solution to the OPF, the Lagrange equation is assumed to satisfy,

$$\nabla \mathscr{L} = 0 \tag{13.118}$$

or, for the state variable, $\mathbf{x}$, we have:

$$\nabla \mathscr{L}_x = \frac{\partial \mathscr{L}}{\partial \mathbf{x}} = \frac{\partial \mathrm{f}}{\partial \mathbf{x}} + \left[\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right]^T \boldsymbol{\lambda} = 0 \tag{13.119}$$

which can be used to solve for $\boldsymbol{\lambda}$ as follows:

$$\left[\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right]^T \boldsymbol{\lambda} = -\frac{\partial \mathrm{f}}{\partial \mathbf{x}} \tag{13.120}$$

The problem here is that the matrix

$$\left[\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right]^T \tag{13.121}$$

has N rows where N equals the number of state variables, and M columns corresponding to M binding constraints. We shall assume that $N \leq M$. The vector

$$\frac{\partial \mathrm{f}}{\partial \mathbf{x}}$$

has N elements and the lambda vector, $\boldsymbol{\lambda}$, has M elements. Thus, the equation which can be used to solve for the lambda vector is overdetermined; that is, there are more elements in the lambda vector than rows in the matrix or the right-hand side. This type of equation has many solutions for the lambda vector. The correct one is found by applying a least-squares technique, as explained in Chapter 12 on state estimation. Further, the usual method of solving for the lambda vector is to apply the QR algorithm (also explained in Chapter 12). Thus, we can use *any method* to solve for the optimal state vector for an OPF and then develop the matrix and right-hand side shown above and solve for the BIC vector.

EXAMPLE 13E

This example gives the bus incremental costs for the same six-bus sample used in Examples 13C and 13D. For the case where both generation MW and

TABLE 13.10 Bus Incremental Costs

Bus	Ɽ/MWh	Ɽ/MVARh
1	12.22	0
2	11.89	0
3	11.97	0.1
4	12.98	0.81
5	12.59	0.51
6	12.29	0.38

scheduled voltages are adjusted to obtain minimum cost, the bus incremental costs are given in Table 13.10.

There is a cost for increasing the MW delivered, as well as the MVAR delivered from or to any bus in the network. In Table 13.10, the bus incremental costs for delivering MW at buses 1, 2, and 3 are equal to the incremental costs of the generator cost functions at the optimal dispatch. The bus incremental cost to deliver MVAR at buses 1 and 2 is zero since these generators are not at their maximum VAR limit and can generate incremental MVAR for "free." The incremental cost to deliver more MVARs at bus 3 is nonzero since generator 3 is at maximum VAR limit and one would have to generate the extra VARs at buses 1 and 2. Finally, the delivery points have higher bus incremental costs since they require that all MW and MVAR consumed at these buses must be delivered via the transmission system, which will cost the system in MW and MVAR losses.

In addition to the bus incremental costs, the procedure outlined above can also be used to generate the cost of changing the limit at any binding constraint. In the case of the dispatch used in Table 13.10, line 2-4 is at an MVA linit and bus 5 is at maximum voltage. The incremental cost with respect to changing the MVA limit on line 2-4 is -1.01 Ɽ/MVAh, indicating that if the limit were increased the system operating cost would decrease. Last of all, the incremental cost of changing the bus 5 upper voltage limit -88.4 Ɽ/pu volt.

PROBLEMS

13.1 You are going to use a linear program and a power flow to solve an OPF. The linear program will be used to solve constrained dispatch problems and the power flow will confirm that you have done the correct thing. For each of the problems, you should use the power flow data for *the six-bus problem found in Chapter 4.*

The following data on unit cost functions applies to this problem:

Unit 1 (bus 1): $F(P) = 600.0 + 6.0P + 0.002P^2$

$P_{min} = 70$ MW

$P_{max} = 250.0$ MW

Unit 2 (bus 2): $F(P) = 220.0 + 7.3P + 0.003P^2$

$P_{min} = 55$ MW

$P_{max} = 135$ MW

Unit 3 (bus 3): $F(P) = 100.0 + 8.0P + 0.004P^2$

$P_{min} = 60$ MW

$P_{max} = 160$ MW

When setting up the LP you should use three straight-line segments with break points as below:

Unit 1, break points at: 70, 130, 180, and 250 MW

Unit 2, break points at: 55, 75, 95, and 135 MW

Unit 3, break points at: 60, 80, 120, and 160 MW

When using the LP for dispatching you should ignore losses.
Set up the power flow as follows:

Load = 300 MW

Generation on bus 2 = 100 MW

Generation on bus 3 = 100 MW

This should lead to a flow of about 67 MW on line 3-6.

Using the linear program, set up a minimum cost LP for the three units using the break points above and the generation shift (or "a") factors from Figure 11.7. You are to constrain the system so that the flow on line 3-6 is no greater than 50 MW.

When you obtain an answer from the LP, enter the values for P_2 and P_3 found in the LP into the load flow and see if, indeed, the flow on line 3-6 is close to the 30 MW desired. (Be sure the load is still set to 300 MW.)

13.2 Using the six-bus power flow example from Chapter 4 with load at 240 MW, try to adjust the MW generated on the three generators and the voltage on each generator to minimize transmission losses. Keep the generators within their economic limits and the voltages at the generators within 0.90 to 1.07 pu volts. Use the following as MVAR limits:

Bus 2 generator: 100 MVAR max

Bus 3 generator: 60 MVAR max

13.3 Using the six-bus power flow example from Chapter 4, set up the base case as in Problem 13.1 (300 MW load, 100 MW on generator buses 2 and 3). Solve the base conditions and note that the load voltages on buses 4, 5, and 6 are quite low. Now, drop the line from bus 2 to bus 3 and resolve the power flow. (Note that the VAR limits on buses 2 and 3 should be the same as in Problem 13.2.)

This results in a severe voltage drop at bus 6. Can you correct this voltage so it comes back into normal range (e.g., 0.90 per unit to 1.07 per unit)? Suggested options: Add fixed capacitance to ground at bus 6, raise the voltage at one or more of the generators, reduce the load MW and MVAR at bus 6, etc.

13.4 You are going to solve the following optimal power flow in two different ways. Given a power system with two generators, P_1 and P_2, with their corresponding cost functions $F_1(P_1)$ and $F_2(P_2)$. In addition, the voltage magnitudes on the generator buses are also to be scheduled.

The balance between load and generation will be assumed to be governed by a linear constraint:

$$\sum_i \beta_i P_i = \sum_i \beta_i P_i^0$$

In addition, two constraints have been identified and their sensitivities calculated. The first is a flow constraint where:

$$\Delta \text{flow}_{nm} = \sum_i af_i \Delta P_i + \sum_i av_i \Delta V_i$$

The second constraint involves a voltage magnitude at bus k which is assumed to be sensitive only to the generator voltages:

$$\Delta V_k = \sum_i \gamma_i \Delta V_i$$

a. Assume that the initial generator outputs are P_1^0 and P_2^0 and that the initial voltage magnitudes are V_1^0 and V_2^0 and that you have obtained the initial flow, flow_{nm}^0, and the initial voltage, V_k^0, from a power flow program.

Further assume that there are limits to be constrained flow and voltage: flow_{nm}^+ and flow_{nm}^- and for the voltage V_k^+ and V_k^-.

Express the flow on line *nm* and the voltage on bus *k* as linear functions of the four control variables: P_1, P_2, V_1, V_2.

b. Show how to obtain the minimum cost with the gradient method. In this case, you may assume that the flow constraint and the voltage constraint are equality constraints where we desire the constraints to be scheduled to the upper limit. Any matrices in this formulation should be shown with all terms; if the inverse is needed, just express it as an inverse matrix—do not try to show all the terms in the inverse itself.

c. Show the same minimun cost dispatch solution with an LP where we break each cost function into two segments.

FURTHER READING

Reference 1 is considered the classic paper that first introduced the concept of an optimal power flow. References 2 and 3 give a good overview of the techniques and methods of OPFs. Reference 4 is a good introduction to the basic mathematics of the gradient method, and references 5–7 cover the Newton OPF method.

Reference 8 shows how the bus incremental costs are calculated using a least-squares approach. Reference 9 is an excellent paper dealing with the application of linear programming to the OPF solution. References 10 and 11 introduce the concept of the interior point algorithm. References 12 and 13 deal with the application of the interior point algorithm to the OPF solution. References 14 and 15 talk extensively about how to incorporate security constraints into the OPF, while reference 16 shows some of the special AGC logic needed when an OPF is holding a line flow constraint.

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