Chapter 5—Practice Examples

P.E. 5.1

$$\nabla^2 V = -\frac{\rho}{\varepsilon} \longrightarrow \frac{d^2V}{dx^2} = -\frac{\rho_o x}{\varepsilon a}$$

$$V = -\frac{\rho_o x^3}{6\varepsilon a} + Ax + B$$

$$\boldsymbol{E} = -\frac{dV}{dx}\boldsymbol{a}_x = \left(\frac{\rho_o x^2}{2\varepsilon a} - A\right)\boldsymbol{a}_x$$

If $\mathbf{E} = 0$ at $x = 0$, then

$$0 = 0 - A \longrightarrow A = 0$$

If $V = 0$ at $x = a$, then

$$0 = -\frac{\rho_o a^3}{6\varepsilon a} + B \longrightarrow B = \frac{\rho_o a^2}{6\varepsilon}$$

Thus

$$V = \frac{\rho_o}{6\varepsilon a}(a^3 - x^3), \qquad \boldsymbol{E} = \frac{\rho_o x^2}{2\varepsilon a}\boldsymbol{a}_x$$

P.E. 5.2

$$V_1 = A_1 x + B_1, \quad V_2 = A_2 x + B_2$$

$$V_1(x = d) = V_o = A_1 d + B_1 \longrightarrow B_1 = V_o - A_1 d$$

$$V_1(x = 0) = 0 = 0 + B_2 \longrightarrow B_2 = 0$$

$$V_1(x = a) = V_2(x = a) \longrightarrow aA_1 + B_1 = A_2 a$$

$$D_{1n} = D_{2n} \longrightarrow \varepsilon_1 A_1 = \varepsilon_2 A_2 \longrightarrow A_2 = \frac{\varepsilon_1}{\varepsilon_2} A_1$$

$$A_1 a + V_o - A_1 d = \frac{\varepsilon_1}{\varepsilon_2} a A_1 \longrightarrow V_o = A_1\left(-a + d + \frac{\varepsilon_1}{\varepsilon_2} a\right)$$

or

$$A_1 = \frac{V_o}{d - a + \varepsilon_1 a / \varepsilon_2}, \quad A_2 = \frac{\varepsilon_1}{\varepsilon_2} A_1 \frac{\varepsilon_1 V_o}{\varepsilon_2 d - \varepsilon_2 a + \varepsilon_1 a}$$

Hence

$$\boldsymbol{E}_1 = -A_1 \boldsymbol{a}_x = \frac{-V_o a_x}{d - a + \varepsilon_1 a / \varepsilon_2}, \qquad E_2 = -A_2 \boldsymbol{a}_x = \frac{-V_o \boldsymbol{a}_x}{a + \varepsilon_2 d / \varepsilon_1 - \varepsilon_2 a / \varepsilon_1}$$

P.E. 5.3

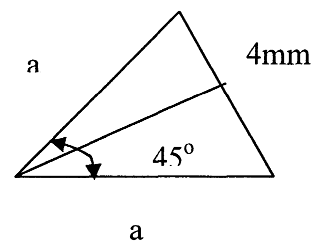
$$\mathbf{E} = -\frac{V_o}{\rho\phi_o}\mathbf{a}_\phi, \qquad D = \varepsilon_o E$$

$$\rho_s = D_n(\phi = 0) = -\frac{V_o\varepsilon}{\rho\phi_o}$$

The charge on the plate $\phi = 0$ is

$$Q = \int \rho_s dS = -\frac{V_o\varepsilon}{\phi_o}\int_{z=0}^{L}\int_{\rho=a}^{b}\frac{1}{\rho}dz d\rho = -\frac{V_o\varepsilon}{\phi_o}L\ln(b/a)$$

$$C = \frac{|Q|}{V_o} = \frac{\varepsilon L}{\phi_o}\ln\frac{b}{a}$$



$$a\sin\frac{45^o}{2} = 2 \quad \longrightarrow \quad a = \frac{2}{\sin 22.5^o} = 5.226 \text{ mm}$$

$$C = \frac{1.5\times\frac{10^{-9}}{36\pi}}{\frac{\pi}{4}}5\ln\frac{1000}{5.226} = 444 \text{ pF}$$

$$Q = CV_o = 444\times10^{-12}\times 50 \text{ C} = \underline{\underline{22.2 \text{ nC}}}$$

P.E. 5.4

$V_o = 50, \quad \theta_1 = 45°, \quad \theta_2 = 90°, \quad r = \sqrt{3^2+4^2+2^2} = \sqrt{29}, \quad \theta = \tan^{-1}\frac{\rho}{z} =$

$\tan^{-1}\frac{5}{2} \quad \longrightarrow \quad \theta = 68.2°$

$$V = \frac{50\ln(\tan 34.1° / \tan 22.5°)}{\ln(\tan 45° / \tan 22.5°)} = \underline{\underline{27.87 \text{ V}}},$$

$$\mathbf{E} = \frac{-50\mathbf{a}_\theta}{\sqrt{29}\sin 68.2°\ln(\tan 45° / \tan 22.5°)} = \underline{\underline{-11.35\mathbf{a}_\theta \text{ V/m}}}$$

P.E. 5.5

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial x}\mathbf{a}_x - \frac{\partial V}{\partial y}\mathbf{a}_y$$

$$= -\frac{4V_o}{b}\sum_{n=\text{odd}}^{\infty}\frac{1}{\sinh n\pi a/b}\left[\cos(n\pi x/b)\sinh(n\pi y/b)\,\mathbf{a}_x + \sin(n\pi x/b)\cosh(n\pi y/b)\mathbf{a}_y\right]$$

(a) At (x,y) = (a, a/2),

$$V = \frac{400}{\pi}(0.3775 - 0.0313 + 0.00394 - 0.000585 + \ldots) = \underline{\underline{44.51\text{ V}}}$$

$$\mathbf{E} = 0\mathbf{a}_x + (-115.12 + 19.127 - 3.9411 + 0.8192 - 0.1703 + 0.035 - 0.0074 + \ldots)\mathbf{a}_y$$
$$= \underline{\underline{-99.25\mathbf{a}_y\text{ V/m}}}$$

(b) At (x,y) = (3a/2, a/4),

$$V = \frac{400}{\pi}(0.1238 + 0.006226 - 0.00383 + 0.0000264 + \ldots) = \underline{\underline{16.50\text{ V}}}$$

$$\mathbf{E} = (24.757 - 3.7358 - 0.3834 + 0.0369 + 0.00351 - 0.00033 + \ldots)\mathbf{a}_x$$
$$+(-66.25 - 4.518 + 0.3988 + 0.03722 - 0.00352 - 0.000333 + \ldots)\mathbf{a}_y$$
$$= 20.68\mathbf{a}_x - 70.34\mathbf{a}_y\text{ V/m}$$

P.E. 5.6

$$V(y=a) = V_o\sin(7\pi x/b) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/b)\sinh(n\pi a/b)$$

By equating coefficients, we notice that $c_n = 0$ for $n \neq 7$. For n=7,

$$V_o\sin(7\pi x/b) = c_7\sin(7\pi x/b)\sinh(7\pi a/b) \longrightarrow c_7 = \frac{V_o}{\sinh(7\pi a/b)}$$

Hence

$$V(x,y) = \underline{\underline{\frac{V_o}{\sinh(7\pi a/b)}\sin(7\pi x/b)\sinh(7\pi y/b)}}$$

P.E. 5.7

Let $V(r,\theta,\phi) = R(r)F(\theta)\Phi(\phi)$.

Substituting this in Laplace's equation gives

$$\frac{\Phi F}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{R\Phi}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dF}{d\theta}\right)+\frac{RF}{r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2}=0$$

Dividing by $RF\Phi / r^2\sin^2\theta$ gives

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2R'\right)+\frac{\sin\theta}{F}\frac{d}{d\theta}(\sin\theta F') = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \lambda^2$$

$$\underline{\underline{\Phi''+\lambda^2\Phi = 0}}$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2R'\right)+\frac{1}{F\sin\theta}\frac{d}{d\theta}(\sin\theta F') = \lambda^2/\sin^2\theta$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2R'\right) = \frac{\lambda^2}{\sin^2\theta}-\frac{1}{F\sin\theta}\frac{d}{d\theta}(\sin\theta F') = \mu^2$$

$$2rR'+r^2R'' = \mu^2R$$

or

$$\underline{\underline{R''+\frac{2}{r}R'-\frac{\mu^2}{r^2}R = 0}}$$

$$\frac{\sin\theta}{F}\frac{d}{d\theta}(\sin\theta F')-\lambda^2+\mu^2\sin^2\theta = 0$$

or

$$F''+\cos\theta F'+(\mu^2\sin\theta-\lambda^2\csc\theta)F = 0$$

P.E. 5.8

(a) This is similar to Example 5.8a except that here ; $0 < \phi < 2\pi$ instead of $0 < \phi < \pi / 2$. Hence

$$I = \frac{2\pi t V_o \sigma}{\ln(b/a)} \quad \text{and} \quad R = \frac{V_o}{I} = \frac{\ln\frac{b}{a}}{\underline{\underline{2\pi t\sigma}}}$$

(b) This to similar to Example 6.8(b) except that here $0 < \phi < 2\pi$. Hence

$$I = \frac{V_o\sigma}{t}\int_a^b\int_0^{2\pi}\rho\, d\rho\, d\phi = \frac{V_o\sigma\pi(b^2 - a^2)}{t}$$

and $R = \frac{V_o}{I} = \frac{t}{\underline{\underline{\sigma\pi(b^2 - a^2)}}}$

P.E. 5.9 From Example 5.9

$$J_1 = \frac{\sigma_1 V_o}{\rho \ln\frac{b}{a}}, \qquad J_2 = \frac{\sigma_2 V_o}{\rho \ln\frac{b}{a}}$$

$$I = \int J \bullet dS = \int_{z=0}^{L}\left[\int_{\phi=0}^{\pi} J_1\rho\, d\phi + \int_{\phi=\pi}^{2\pi} J_2\rho\, d\phi\right]dz = \frac{V_o l}{\ln\frac{b}{a}}\left[\pi\sigma_1 + \pi\sigma_2\right]$$

$$R = \frac{V_o}{I} = \frac{\ln\frac{b}{a}}{\pi l\left[\sigma_1 + \sigma_2\right]}$$

P.E. 5.10

(a) $C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$, C_1 and C_2 are in series.

$$C_1 = 4\pi x \frac{10^{-9}}{36\pi}\left(\frac{2.5}{\frac{10^3}{2} - \frac{10^3}{3}}\right) = 5/3 \text{ pF}, \qquad C_2 = 4\pi x \frac{10^{-9}}{36\pi}\left(\frac{3.5}{\frac{10^3}{1} - \frac{10^3}{2}}\right) = 7/9 \text{ pF}$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{(5/3)(7/9)}{(5/3) + (7/9)} = \underline{\underline{0.53 \text{ pF}}}$$

(b) $C = \frac{2\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$, C_1 and C_2 are in parallel.

$$C_1 = 2\pi x \frac{10^{-9}}{36\pi}\left(\frac{2.5}{\frac{10^3}{1} - \frac{10^3}{3}}\right) = 5/24 \text{ pF}, \qquad C_2 = 2\pi x \frac{10^{-9}}{36\pi}\left(\frac{3.5}{\frac{10^3}{1} - \frac{10^3}{3}}\right) = 7/24 \text{ pF}$$

$$C = C_1 + C_2 = \underline{\underline{0.5 \text{ pF}}}$$

P.E. 5.11

As in Example 5.8, assuming $V(\rho = a) = 0, \quad V(\rho = b) = V_o,$

$$V = V_o \frac{\ln \rho / a}{\ln b / a}, \qquad E = -\nabla V = -\frac{V_o}{\rho \ln b / a} a_\rho$$

$$Q = \int \varepsilon E \bullet dS = \frac{V_o \varepsilon}{\ln b / a} \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \frac{1}{\rho} dz \rho d\phi = \frac{V_o 2\pi\varepsilon L}{\ln b / a}$$

$$C = \frac{Q}{V_o} = \underline{\underline{\frac{2\pi\varepsilon L}{\ln b / a}}}$$

P.E. 5.12

(a) Let C_1 and C_2 be capacitances per unit length of each section and C_T be the total capacitance of 10m length. C_1 and C_2 are in series.

$$C_1 = \frac{2\pi\varepsilon_{r1}\varepsilon_o}{\ln b/c} = \frac{2\pi x 2.5}{\ln 3/2}\frac{10^{-9}}{36\pi} = 342.54 \text{ pF/m,}$$

$$C_2 = \frac{2\pi\varepsilon_{r2}\varepsilon_o}{\ln c/a} = \frac{2\pi x 3.5}{\ln 2}\frac{10^{-9}}{36\pi} = 280.52 \text{ pF/m}$$

$$C = \frac{C_1C_2}{C_1+C_2} = \frac{342.54x280.52}{342.54+280.52} = 154.22 \text{ pF}$$

$$C_T = Cl = \underline{\underline{1.54}} \text{ nF}$$

(b) C_1 and C_2 are in parallel.

$$C = C_1 + C_2 = \frac{\pi\varepsilon_{r1}\varepsilon_o}{\ln b/a} + \frac{\pi\varepsilon_{r2}\varepsilon_o}{\ln b/a} = \frac{\pi(\varepsilon_{r1}+\varepsilon_{r2})\varepsilon_o}{\ln b/a} = \frac{6\pi}{\ln 3}\frac{10^{-9}}{36\pi} = 151.7 \text{ pF/m}$$

$$C_T = Cl = \underline{\underline{1.52}} \text{ nF}$$

P.E. 5.13

Instead of Eq. (5.31), we now have

$$V = -\int_b^a \frac{Qdr}{4\pi\varepsilon r^2} = -\int_b^a \frac{Qdr}{4\pi\frac{10\varepsilon_o}{r}r^2} = -\frac{Q}{40\pi\varepsilon_o}\ln b/a$$

$$C = \frac{Q}{|V|} = \frac{40\pi}{\ln 4/1.5}\frac{10^{-9}}{36\pi} = \underline{\underline{1.13 \text{ nF}}}$$